65. On an Invariant of Veronesean Rings

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§ 1. **Main result.** Let K be a field and $t_1, \cdots, t_n$ indeterminates. Let m be a positive integer. In this paper we consider the ring $R_{n,m}$ generated, over K, by all the monomials $t_1^{p_1}\cdots t_n^{p_n}$ such that $\sum_{i=1}^{n} p_i = m$. Let $S_{n,m}$ be the localization of $R_{n,m}$ at the maximal ideal generated by all $t_1^{p_1}\cdots t_n^{p_n}$ in $R_{n,m}$. In [2] Gröbner showed that the local ring $S_{n,m}$ is a Macaulay ring of dimension n. In this paper this ring is called a *Veronesean* local ring.

In general, it is well known that in a Macaulay local ring the number of the irreducible components of an ideal generated by a system of parameters is an invariant of the ring. This invariant is called the *type* of the ring (cf. [4]). A Macaulay local ring is a Gorenstein ring if and only if the ring has type one.

The aim of this paper is to prove the following theorem.

Theorem. *Let $S_{n,m}$ be a Veronesean local ring. Then*

$$\text{type}\, S_{n,m} = 1 \qquad \text{if } n \equiv 0 \ (\text{mod. } m)$$

and

$$\text{type}\, S_{n,m} = \binom{n+m-r-1}{n-1} \quad \text{if } n \equiv r \ (\text{mod. } m) \quad 0 < r < m.$$

As a direct consequence of the theorem, we have the following

Corollary. *A Veronesean local ring $S_{n,m}$ is a Gorenstein ring if and only if $n=1$ or $n \equiv 0$ (mod. m).*

§ 2. **Proof of theorem.** For a non-negative integer s, we denote by P(s) the set of ordered n-tuples $(p)=(p_1, \cdots, p_n)$ of non-negative integers p_i such that $\sum_{i=1}^{n} p_i = sm$. We also denote by $t^{(p)}$ the monomial $t_1^{p_1}\cdots t_n^{p_n}$. With the same notation as in § 1, the ring $R_{n,m}=K[t^{(p)} \mid (p) \in P(1)]$. Let $\mathfrak{m}$ be the maximal ideal generated by all $t^{(p)}$, $(p) \in$ P(1), and $\mathfrak{q}$ the ideal generated by $t_1^m, \cdots, t_n^m$. Then $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal. Since the localization $S_{n,m}$ of $R_{n,m}$ at $\mathfrak{m}$ is a Macaulay local ring of dimension n and since $\{t_1^m, \cdots, t_n^m\}$ is a maximal regular sequence of $S_{n,m}$ (cf. [2]), the type of $S_{n,m}$ is given by the dimension of the K-vector space $(\mathfrak{q} : \mathfrak{m})/\mathfrak{q}$ (cf. [4]).

Before proving some lemmas we give preliminary remarks: A monomial $t^{(p)}$ is in $R_{n,m}$ if and only if (p) is in P(s) for some s. If (p)

is in P(s), then $t^{(p)}$ is in $\mathfrak{m}^s$. The ideal $\mathfrak{m}^s$ is generated by all $t^{(p)}$, (p) $\in$ P(s). Let Q(s) be the set consisting of all (p) in P(s) such that $p_i<m$ for $1\leq i\leq n$. Let (p) be in P(s). Then $t^{(p)}$ is in $\mathfrak{q}$ if and only if (p) is not in Q(s). Hence $\mathfrak{m}^s\subseteq\mathfrak{q}$ if and only if Q(s) is the empty set.

Lemma 1. *Assume that $n\geq 2$ and $m\geq 2$. Let k be the integer such that $n-n/m<k\leq n-n/m+1$. Then $\mathfrak{m}^k\subseteq\mathfrak{q}$ and $\mathfrak{m}^{k-1}\not\subseteq\mathfrak{q}$.*

Proof. Let (p) be in P(k). Then $\sum_{i=1}^{n} p_i=km>(m-1)n$. Hence $p_j\geq m$ for some j. This shows that $\mathfrak{m}^k\subseteq\mathfrak{q}$. Next we show that $\mathfrak{m}^{k-1}\not\subseteq\mathfrak{q}$. In order to prove this it is enough to show that Q($k-1$) is not the empty set. First we consider the case when $n\geq m$. Let $d=(m-1)n-(k-1)m$. Then d is a non-negative integer. Since $n\geq m$ and $km-(m-1)n>0$, we have $n-d=n-m+km-(m-1)n>0$. If $d=0$, we set $p_i=m-1$ for $1\leq i\leq n$. If $d>0$, we set $p_i=m-2$ for $1\leq i\leq d$ and $p_i=m-1$ for $d+1\leq i\leq n$. Then (p) is in Q($k-1$). Next we consider the case when $m>n$. In this case we have $k=n$. Let $m=qn+w$, $0\leq w<n$. Set $p_1=(n-1)q$ and $p_i=(n-1)q+w$ for $2\leq i\leq n$. Then (p) is in Q($k-1$). Hence in any case Q($k-1$) is not the empty set. q.e.d.

We remark that if $n\geq 2$ and $m\geq 2$, then the integer k in Lemma 1 is not less than 2.

Lemma 2. *Assume that $n\geq 2$ and $m\geq 2$. Let k be the same integer as in Lemma 1. If $s\leq k-2$, then for each (p) in Q(s) there exists (u) in P(1) such that $p_i+u_i<m$ for $1\leq i\leq n$.*

Proof. Set $q_i=m-p_i$. Then $0<q_i\leq m$ and $\sum_{i=1}^{n}(q_i-1)=(n-s)m-n\geq(n-k+2)m-n\geq m$. Hence we can choose integers u_i so that $q_i-1\geq u_i\geq 0$ and $\sum_{i=1}^{n} u_i=m$. Then $p_i+u_i<m$. q.e.d.

Lemma 3. *Assume that $n\geq 2$ and $m\geq 2$. Let k be the same integer as in Lemma 1. Then $\mathfrak{q}:\mathfrak{m}=\mathfrak{q}+\mathfrak{m}^{k-1}$.*

Proof. Since $\mathfrak{m}^k\subseteq\mathfrak{q}$ by Lemma 1, we have $\mathfrak{q}+\mathfrak{m}^{k-1}\subseteq\mathfrak{q}:\mathfrak{m}$. We show the opposite inclusion. Let x be an element in $\mathfrak{q}:\mathfrak{m}$. We can write $x=\sum a_{(p)}t^{(p)}+y$, where y is an element in $\mathfrak{q}+\mathfrak{m}^{k-1}$, $a_{(p)}$ are elements in K and the sum $\sum$ is taken for all (p) in $\mathrm{Q}=\bigcup_{j=0}^{k-2}\mathrm{Q}(j)$. We show that $a_{(p)}=0$ for all (p) in Q. Let (q) be in Q. Then by Lemma 2 there exists (v) in P(1) such that $q_i+v_i<m$ for $1\leq i\leq n$. Let Q′ be the set consisting of all (p) in Q such that $p_i+v_i<m$ for $1\leq i\leq n$. Since $x\mathfrak{m}\subseteq\mathfrak{q}$ and $y\mathfrak{m}\subseteq\mathfrak{q}$ by Lemma 1, $\sum' a_{(p)}\,t^{(p+v)}$ is in $\mathfrak{q}$, where the sum $\sum'$ is taken for all (p) in Q′. Therefore we have $a_{(p)}=0$ for all (p) in Q′, and hence $a_{(q)}=0$. This shows that x is in $\mathfrak{q}+\mathfrak{m}^{k-1}$. q.e.d.

Before proving the theorem, we remark that if $\mathfrak{m}^{h+1}\subseteq\mathfrak{q}$, then the dimension of the K-vector space $(\mathfrak{q}+\mathfrak{m}^h)/\mathfrak{q}$ is equal to the number of elements of Q(h).

Proof of theorem. For $n=1$ or $m=1$, $S_{n,m}$ is a regular local ring, hence it is a Gorenstein ring, that is, type $S_{n,m}=1$. Therefore it is enough to prove the theorem for $n\geq 2$ and $m\geq 2$. In case when $n\equiv 0$ (mod. m): Let $n=mq$. Then the integer k in Lemma 1 is equal to $(m-1)q+1$. Since $\sum_{i=1}^{n} p_i=(m-1)qm=(m-1)n$ for (p) in $\mathrm{P}(k-1)$, $\mathrm{Q}(k-1)$ consists of only one tuple $(m-1, \cdots, m-1)$. Hence by Lemma 3 we have type $S_{n,m}=1$. In case when $n\equiv r$ (mod. m) $0<r<m$: Let $n=mq+r$. Then $k=(m-1)q+r$. Let Q' be the set of n-tuples $(q)=(q_1, \cdots, q_n)$ such that $q_i\geq 0$ for $1\leq i\leq n$ and $\sum_{i=1}^{n} q_i=m-r$. Since $\sum_{i=1}^{n}(m-1-p_i)=m-r$ for every (p) in $\mathrm{Q}(k-1)$, the map $\mathrm{Q}(k-1)\to \mathrm{Q}'$ defined by $(p)\mapsto(q)$, $q_i=m-1-p_i$, is a bijection. Hence type $S_{n,m}$ is equal to the number of elements of Q'. Obviously it is equal to $\binom{n+m-r-1}{n-1}$. q.e.d.

Remark. If the ground field K has characteristic zero, $R_{n,m}$ is the ring of invariants of a cyclic group of order m acting on $K[t_1, \cdots, t_n]$. In this case, our results are closely related to K. Watanabe [5] and [6].*)

§ 3. **Supplementary results.** In this section we give some results on the connection between the type, the embedding dimension and the dimension of a Veronesean local ring. Let T be the polynomial ring over K, in $\binom{n+m-1}{n-1}$ indeterminates $X_{(p)}$, $(p)\in \mathrm{P}(1)$. Let $\phi: T\to R_{n,m}$ be the ring homomorphism defined by $\phi(X_{(p)})=t^{(p)}$. Let S be the localization of T at the maximal ideal of T generated by all $X_{(p)}$, $(p)\in \mathrm{P}(1)$. Since the kernel of ϕ is generated by all $X_{(p)}X_{(q)}-X_{(u)}X_{(v)}$, $p_i+q_i=u_i+v_i$ for $1\leq i\leq n$ (cf. [2]), the local homomorphism $\psi: S\to S_{n,m}$ induced by ϕ is a minimal embedding of $S_{n,m}$, that is, the kernel of ψ is contained in the square of the maximal ideal of S. Hence the embedding dimension of $S_{n,m}$ is equal to $\binom{n+m-1}{n-1}$. We first note that $S_{n,m}$ is a regular local ring if and only if $n=1$ or $m=1$. This follows from the fact that $S_{n,m}$ is regular if and only if $\binom{n+m-1}{n-1}=n$. In [2] Gröbner showed that the kernel of ϕ, and hence the kernel of ψ, are minimally generated by $c=\binom{e+1}{2}-\binom{2m+n-1}{n-1}$ elements, where e is the embedding dimension of $S_{n,m}$, that is, $e=\binom{n+m-1}{n-1}$. Hence $S_{n,m}$

*) Especially, in the characteristic zero case, the theorem in §1 is an easy consequence of Lemma 6 in [5] or of Lemma 7 in [6]. In the positive characteristic case, however, the theorem is not contained in [5] and [6].

is a complete intersection if and only if $c=e-n$. We now show the following

Proposition 1. *A Veronesean local ring $S_{n,m}$ which is not a regular local ring is a complete intersection if and only if $n=m=2$.*

Proof. If $n=m=2$, then $c=e-n=1$. Hence $S_{2,2}$ is a complete intersection. Conversely assume that $(n, m)\neq(2,2)$. By the corollary in §1 we may, furthermore, assume that $n=mq$ for some positive integer q. Write $\binom{2m+n-1}{n-1}=de$, where $d=\prod_{i=1}^{m}(2m+n-i)/(2m+1-i)$. Since $(n+m-i)/(m+1-i)>(2m+n-i)/(2m+1-i)$ for $1\leq i\leq m-1$ and since $n-2(n+m)/(m+1)=m\{q(m-1)-2\}/(m+1)\geq 0$, we have $e-2d>0$. Therefore we have $c-e+n=(e/2)(e-2d-1)+n>0$. This shows that $S_{n,m}$ is not a complete intersection. q.e.d.

If $n\geq 3$ and $m\geq 2$ and if $n\equiv 0$ (mod. m), then $S_{n,m}$ is an example of an n-dimensional normal Gorenstein local domain which is not a complete intersection.

Proposition 2. *If a Veronesean local ring $S_{n,m}$ is not a regular local ring, then the following inequality holds;*

$$\operatorname{emdim} S_{n,m}-\dim S_{n,m}\geq \operatorname{type} S_{n,m}.$$

Proof. Since $\operatorname{emdim} S_{n,m}-\dim S_{n,m}>0$, the inequality obviously holds when $n\equiv 0$ (mod. m). Consider the case when $n\equiv r$ (mod. m) $0<r<m$. Since, in general, $\binom{s+1}{t+1}=\sum_{i=t}^{s}\binom{i}{t}$, we have $\binom{n+m-1}{n-1}=\binom{n+m-r-1}{n-1}+h$, where $h=\sum_{i=1}^{r}\binom{n+m-i-1}{n-2}$. If $n=2$, then $r=2$ and $m>2$. Hence $h=n=2$. If $n>2$, then $h\geq\binom{n+m-r-1}{n-2}\geq\binom{n-1}{n-2}+1=n$. Therefore we have $\binom{n+m-1}{n-1}-n\geq\binom{n+m-r-1}{n-1}$ for $n\geq 2$ and $m\geq 2$, and this is the required inequality. q.e.d.

Remark. In general, for a Macaulay local ring R, the following inequalities hold: (1) multiplicity $R\geq\operatorname{emdim} R-\dim R+1$ (Abhyankar 1]); (2) multiplicity $R\geq\operatorname{type} R+1$ if R is not regular (Engelken, cf. [3]). For a Macaulay local ring R which is not regular, the inequality $\operatorname{emdim} R-\dim R\geq\operatorname{type} R$ does not hold in general. In fact, consider the ring $R=K[X, Y]/(X, Y)^t$, $t\geq 2$. Then R is a Macaulay local ring of dimension zero, and has embedding dimension 2 and type t. Hence, for $t\geq 3$ the inequality does not hold.

References

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