80. *The Completion by Cuts of an M-symmetric Lattice*

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It is well known that the completion by cuts of a modular lattice is not necessarily modular ([1], p. 127, Example 9). But the following question was open ([2], p. 55, Problem 4): Is the completion by cuts of an M-symmetric lattice M-symmetric? In this paper we will give a negative answer to this question by constructing an atomistic M-symmetric lattice whose completion by cuts is not M-symmetric.

Let E be an infinite set and let A, B, C, D be mutually disjoint subsets of E which are all infinite. We take a sequence of subsets $\{C_n\}$ of C which satisfies the following two conditions:

(1) $C=C_0\supset C_1\supset C_2\supset\cdots$ and $\bigcap_{n=1}^{\infty} C_n=\phi$ (empty).

(2) For every $n=1,2,\cdots$, the set $C_{n-1}-C_n$ is infinite.

Moreover, we take a sequence of subsets $\{D_n\}$ of D satisfying the same conditions, and we put $A_n=A\cup C_n$ and $B_n=B\cup D_n$. We denote by $\boldsymbol{F}$ the family of all finite subsets of E, and we put

$$\boldsymbol{L}=\{E, A_n\cup F, B_n\cup F, F;\ 1\leqq n<\infty, F\in \boldsymbol{F}\}.$$

Proposition 1. *$\boldsymbol{L}$ forms an atomistic M-symmetric lattice, ordered by set-inclusion.*

Proof. It is evident that if $X, Y\in \boldsymbol{L}$ then their intersection $X\cap Y$ belongs to $\boldsymbol{L}$. Hence, the meet $X\wedge Y$ exists and equals to $X\cap Y$. If $X=A_m\cup F_1$ and $Y=B_n\cup F_2$ $(F_1, F_2\in \boldsymbol{F})$, then since E is the only upper bound of $\{X, Y\}$ in $\boldsymbol{L}$, the join $X\vee Y$ is E. Hence, $X\vee Y$ exists for every $X, Y\in \boldsymbol{L}$ and it holds that

(3) $X\vee Y=\begin{cases} X\cup Y & \text{if } X\cup Y\in \boldsymbol{L} \\ E & \text{if } X\cup Y\notin \boldsymbol{L}.\end{cases}$

Thus, $\boldsymbol{L}$ is a lattice and evidently it is atomistic. Next, we shall show that

(4) $(X, Y)M$ in $\boldsymbol{L}$ if and only if $X\cup Y\in \boldsymbol{L}$.

($(X, Y)M$ means that the pair (X, Y) is modular. See [2], (1.1).) If $X\neq E, Y\neq E$ and $X\cup Y\in \boldsymbol{L}$, then for any $X_1, Y_1\in \boldsymbol{L}$ with $X_1\leqq X$ and $Y_1\leqq Y$ we have $X_1\cup Y_1\in \boldsymbol{L}$. Hence, if $Y_1\leqq Y$ in $\boldsymbol{L}$, then

$$(Y_1\vee X)\wedge Y=(Y_1\cup X)\cap Y=Y_1\cup(X\cap Y)=Y_1\vee(X\wedge Y).$$

Hence, $(X, Y)M$. To prove the converse, it suffices to show that if $X=A_m\cup F_1$, $Y=B_n\cup F_2$ then the pairs (X, Y) and (Y, X) are not modular. Put $Y_1=B_{n+1}$. Then $Y_1\leqq Y$, and since $Y_1\vee X=E$ by (3) we have $(Y_1 \vee X) \wedge Y = Y$. On the other hand, since $X \cap Y$ is finite and since $Y - Y_1 = (B \cup D_n \cup F_2) - (B \cup D_{n+1}) \supset D_n - D_{n+1}$ is infinite, we have $Y_1 \vee (X \wedge Y) = Y_1 \cup (X \cap Y) \neq Y$. Hence, (X, Y) is not modular. Similarly, it holds that (Y, X) is not modular. Thus (4) has been proved, and hence $\boldsymbol{L}$ is M-symmetric.

Following [2], (12.1), for any subset X of $\boldsymbol{L}$ we denote by $\boldsymbol{X}^u$ (resp. $\boldsymbol{X}^l$) the set of upper bounds (resp. lower bounds) of $\boldsymbol{X}$. The completion by cuts of $\boldsymbol{L}$, which is the family $\{\boldsymbol{X} \subset \boldsymbol{L}; \boldsymbol{X} = \boldsymbol{X}^{ul}\}$, is denoted by $\bar{\boldsymbol{L}}$.

Lemma. *For any subset S of E, we put $\boldsymbol{J}(S) = \{X \in \boldsymbol{L}; X \subset S\}$.*

(i) *If $X \in \boldsymbol{L}$ then $\boldsymbol{J}(X) \in \bar{\boldsymbol{L}}$.*

(ii) *$\boldsymbol{J}(S)^u = \{X \in \boldsymbol{L}; X \supset S\}$ for every $S \subset E$.*

(iii) *If $\boldsymbol{J}(S_1), \boldsymbol{J}(S_2) \in \bar{\boldsymbol{L}}$ then $\boldsymbol{J}(S_1) \wedge \boldsymbol{J}(S_2) = \boldsymbol{J}(S_1 \cap S_2)$ in $\bar{\boldsymbol{L}}$. If moreover $\boldsymbol{J}(S_1 \cup S_2) \in \bar{\boldsymbol{L}}$ then $\boldsymbol{J}(S_1) \vee \boldsymbol{J}(S_2) = \boldsymbol{J}(S_1 \cup S_2)$.*

(iv) *$\boldsymbol{J}(A \cup F), \boldsymbol{J}(B \cup F) \in \bar{\boldsymbol{L}}$ for every $F \in \boldsymbol{F}$; especially, $\boldsymbol{J}(A), \boldsymbol{J}(B) \in \bar{\boldsymbol{L}}$.*

(v) *If $\boldsymbol{X} < \boldsymbol{J}(A)$ (resp. $\boldsymbol{X} < \boldsymbol{J}(B)$) in $\bar{\boldsymbol{L}}$ then $\boldsymbol{X} = \boldsymbol{J}(F)$ for some $F \in \boldsymbol{F}$ with $F \subset A$ (resp. $F \subset B$).*

Proof. (i) is evident.

(ii) Let $X \in \boldsymbol{J}(S)^u$. For every $x \in S$, we have $\{x\} \in \boldsymbol{J}(S)$, since $\{x\} \in \boldsymbol{F} \subset \boldsymbol{L}$. Hence, $\{x\} \leqq X$, i.e. $x \in X$. Therefore, $X \supset S$. The converse is evident.

(iii) If $\boldsymbol{J}(S_1), \boldsymbol{J}(S_2) \in \bar{\boldsymbol{L}}$, then since $\boldsymbol{X} \wedge \boldsymbol{Y} = \boldsymbol{X} \cap \boldsymbol{Y}$ for every $\boldsymbol{X}, \boldsymbol{Y} \in \bar{\boldsymbol{L}}$, we have $\boldsymbol{J}(S_1) \wedge \boldsymbol{J}(S_2) = \boldsymbol{J}(S_1) \cap \boldsymbol{J}(S_2) = \boldsymbol{J}(S_1 \cap S_2)$. Moreover, we have $(\boldsymbol{J}(S_1) \cup \boldsymbol{J}(S_2))^u = \boldsymbol{J}(S_1)^u \cap \boldsymbol{J}(S_2)^u = \{X \in \boldsymbol{L};\ X \supset S_1 \cup S_2\} = \boldsymbol{J}(S_1 \cup S_2)^u$ by (ii). Hence, if $\boldsymbol{J}(S_1 \cup S_2) \in \bar{\boldsymbol{L}}$, we have $\boldsymbol{J}(S_1) \vee \boldsymbol{J}(S_2) = (\boldsymbol{J}(S_1) \cup \boldsymbol{J}(S_2))^{ul} = \boldsymbol{J}(S_1 \cup S_2)^{ul} = \boldsymbol{J}(S_1 \cup S_2)$.

(iv) If $X \in \boldsymbol{J}(A \cup F)^{ul}$, then since $A_n \cup F \in \boldsymbol{J}(A \cup F)^u$ for every n, we have $X \subset \bigcap_n (A_n \cup F) = A \cup F$, whence $X \in \boldsymbol{J}(A \cup F)$. Therefore, $\boldsymbol{J}(A \cup F) = \boldsymbol{J}(A \cup F)^{ul} \in \bar{\boldsymbol{L}}$. Similarly, $\boldsymbol{J}(B \cup F) \in \bar{\boldsymbol{L}}$.

(v) Let $\boldsymbol{X} < \boldsymbol{J}(A)$ in $\bar{\boldsymbol{L}}$. Since $\boldsymbol{X}^u \supsetneqq \boldsymbol{J}(A)^u$, there exists $X \in \boldsymbol{X}^u$ with $X \notin \boldsymbol{J}(A)^u$. Since $X \in \boldsymbol{L}$ and $X \not\supset A$, it is easily seen that $X \cap A_1 \in \boldsymbol{F}$. Since $A_1 \in \boldsymbol{J}(A)^u \subset \boldsymbol{X}^u$, we have $X \cap A_1 \in \boldsymbol{X}^u$. Therefore, $\boldsymbol{X}^u$ is a dual ideal of $\boldsymbol{L}$ containing a finite subset. Hence, there exists the smallest finite subset F contained in $\boldsymbol{X}^u$, and then $\boldsymbol{X}^u = \{X \in \boldsymbol{L}; X \supset F\}$. Therefore, $\boldsymbol{X} = \boldsymbol{X}^{ul} = \{X \in \boldsymbol{L}; X \subset F\} = \boldsymbol{J}(F)$. Evidently, $F \subset A$.

Proposition 2. *$\bar{\boldsymbol{L}}$ is not M-symmetric.*

Proof. We shall show that $(\boldsymbol{J}(B \cup F), \boldsymbol{J}(A))M$ in $\bar{\boldsymbol{L}}$ for every $F \in \boldsymbol{F}$. If $\boldsymbol{X} < \boldsymbol{J}(A)$, then it follows from (v) of Lemma that $\boldsymbol{X} = \boldsymbol{J}(F_0)$ with $F_0 \in \boldsymbol{F}, F_0 \subset A$. Hence, by (iv) and (iii) of Lemma, we have $(\boldsymbol{X} \vee \boldsymbol{J}(B \cup F)) \wedge \boldsymbol{J}(A) = \boldsymbol{J}(B \cup F \cup F_0) \wedge \boldsymbol{J}(A) = \boldsymbol{J}((F \cap A) \cup F_0) = \boldsymbol{J}(F_0) \vee \boldsymbol{J}(F \cap A) = \boldsymbol{X} \vee (\boldsymbol{J}(B \cup F) \wedge \boldsymbol{J}(A))$. Therefore, $(\boldsymbol{J}(B \cup F), \boldsymbol{J}(A))M$.

Next, we shall show that if $\phi \neq F \in \boldsymbol{F}$ and $F \cap (A \cup B) = \phi$ then the pair $(\boldsymbol{J}(A), \boldsymbol{J}(B \cup F))$ is not modular. We have $\boldsymbol{J}(B) < \boldsymbol{J}(B \cup F)$ since

$F \notin \boldsymbol{J}(B)$. Since $(\boldsymbol{J}(A) \cup \boldsymbol{J}(B))^u = \boldsymbol{J}(A)^u \cap \boldsymbol{J}(B)^u = \{X \in \boldsymbol{L}; X \supset A \cup B\} = \{E\}$, we have $\boldsymbol{J}(A) \vee \boldsymbol{J}(B) = (\boldsymbol{J}(A) \cup \boldsymbol{J}(B))^{ul} = \boldsymbol{L}$. Hence, $(\boldsymbol{J}(B) \vee \boldsymbol{J}(A)) \wedge \boldsymbol{J}(B \cup F) = \boldsymbol{J}(B \cup F)$. On the other hand, $\boldsymbol{J}(B) \vee (\boldsymbol{J}(A) \wedge \boldsymbol{J}(B \cup F)) = \boldsymbol{J}(B) \vee \boldsymbol{J}(\phi) = \boldsymbol{J}(B)$. Therefore, $(\boldsymbol{J}(A), \boldsymbol{J}(B \cup F))$ is not modular.

Remark 1. (i) By the proof of Proposition 2, $\bar{\boldsymbol{L}}$ is not $\perp$-symmetric ([2], Definition (1.11)).

(ii) A pair (X, Y) in $\boldsymbol{L}$ is dual modular if and only if $X \cup Y \in \boldsymbol{L}$. Indeed, if $X \cup Y \in \boldsymbol{L}$, then for any $Y_1 \geqq Y$ we have $Y_1 \wedge (X \vee Y) = Y_1 \cap (X \cup Y) = (Y_1 \cap X) \cup Y = (Y_1 \wedge X) \vee Y$, whence (X, Y) is dual modular. If $X = A_m \cup F_1$ and $Y = B_n \cup F_2$, then since $X \cup Y \neq E$, we can take $x \in E - (X \cup Y)$. Putting $Y_1 = Y \cup \{x\}$, we have $Y_1 \wedge (X \vee Y) = Y_1 \wedge E = Y_1 \ni x$. But, since $Y_1 \wedge X$ is a finite set, $(Y_1 \wedge X) \vee Y = (Y_1 \cap X) \cup Y \not\ni x$. Hence, (X, Y) is not dual modular.

From this fact, $\boldsymbol{L}$ is M*-symmetric and hence it is finite-modular ([2], (9.5)). Moreover, together with (4), $\boldsymbol{L}$ is cross-symmetric and dual cross-symmetric ([2], (1.9)).

(iii) It follows from (ii) and [2], (12.7) that $\bar{\boldsymbol{L}}$ is a finite-modular AC-lattice. This is an example on Problem 2 in [2].

Remark 2. Though Problems 2 and 3 were solved affirmatively by M. F. Janowitz, we give here a new simple example of an AC-lattice which is neither M-symmetric nor ∇-symmetric (∇-symmetry means that $a \nabla b$ implies $b \nabla a$).

Let E be an infinite set and let $a, b \in E (a \neq b)$. We put $A = E - \{a, b\}$ and

$$\boldsymbol{L} = \{E, A\} \cup \boldsymbol{F} \ (\boldsymbol{F} \text{ is the set of all finite subsets of } E).$$

Evidently, $\boldsymbol{L}$ is a complete lattice by set inclusion, where the meet of elements of $\boldsymbol{L}$ coincides with their intersection, and $A \vee \{a\} = A \vee \{b\} = E$. It is easily verified that $\boldsymbol{L}$ is an AC-lattice. The pair $(\{a, b\}, A)$ is evidently modular. But, $(A, \{a, b\})$ is not modular, since $(\{a\} \vee A) \wedge \{a, b\} = E \wedge \{a, b\} = \{a, b\} \neq \{a\} = \{a\} \vee (A \wedge \{a, b\})$. Moreover, $\{a\} \nabla A$ holds evidently, but $A \nabla \{a\}$ does not hold, since $(\{b\} \vee A) \wedge \{a\} = \{a\} \neq \{b\} \wedge \{a\}$.

References

[1] G. Birkhoff: Lattice Theory (3rd edition). Amer. Math. Soc. Colloq. Publ., Providence (1967).

[2] F. Maeda and S. Maeda: Theory of Symmetric Lattices. Springer, Berlin-Heidelberg-New York (1970).