77. *Oscillation Theorems for Second Order Differential Equations with Retarded Argument*

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Introduction. In this paper we are concerned with the oscillatory behavior of solutions of the differential equation with retarded argument

(A) $$(r(t)x'(t))' + a(t)f(x(g(t))) = 0,$$

where the following conditions are always assumed to hold:

(a) $r(t) \in C^1(0, \infty)$, $r(t) > 0$;

(b) $a(t) \in C(0, \infty)$, $a(t) \geqq 0$;

(c) $g(t) \in C^1(0, \infty)$, $g(t) \leqq t$, $g'(t) \geqq 0$, $\lim_{t \to \infty} g(t) = \infty$;

(d) $f(y) \in C(-\infty, \infty) \cap C^1(-\infty, 0) \cap C^1(0, \infty)$, $yf(y) > 0$, $f'(y) \geqq 0$ for $y \neq 0$.

We consider only those solutions of (A) which are defined and nontrivial for all sufficiently large t. Such a solution is called oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

Our purpose here is to present criteria (sufficient conditions) for all solutions of (A) to be oscillatory not only for the case $\int^{\infty} \frac{dt}{r(t)} = \infty$ but also for the case $\int^{\infty} \frac{dt}{r(t)} < \infty$. Our theorems can be applied to produce oscillation criteria for the damped equation

(B) $$x''(t) + p(t)x'(t) + q(t)f(x(g(t))) = 0.$$

1. We begin with the case $\int^{\infty} \frac{dt}{r(t)} = \infty$. In this case the following theorem holds.

Theorem 1. *Assume there exist two positive functions* $\rho(t) \in C^2(0, \infty)$ *and* $\phi(y) \in C^1(0, \infty)$ *with the following properties:*

$$\rho'(t) \geqq 0, \quad (r(t)\rho'(t))' \leqq 0, \quad \phi'(y) \geqq 0,$$

$$\int_{\pm\delta}^{\pm\infty} \frac{dy}{f(y)\phi(y)} < \infty \qquad \textit{for some } \delta > 0,$$

$$\int^{\infty} \frac{\rho(g(t))a(t)}{\phi(R_T(g(t)))} dt = \infty \qquad \textit{for any } T > 0,$$

where $R_T(t) = \int_T^t \frac{ds}{r(s)}$. *Then all solutions of* (A) *are oscillatory.*

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Proof. Suppose there exists a nonoscillatory solution $x(t)$ of (A). Without loss of generality we may assume that $x(g(t))>0$ for all sufficiently large t, say, $t \geqq T$. From (A) $(r(t)x'(t))' = -a(t)f(x(g(t))) \leqq 0$, which implies that $r(t)x'(t)$ is nonincreasing. From the assumption $\int^{\infty} \frac{dt}{r(t)} = \infty$ it follows that $x'(t) \geqq 0$, i.e., $x(t)$ is nondecreasing for $t \geqq T$. In fact, if $x'(t^*)<0$ for some $t^* \geqq T$, then $r(t)x'(t) \leqq r(t^*)x'(t^*)$ for $t \geqq t^*$, and an integration of the last inequality divided by $r(t)$ gives

$$x(t)-x(^*) \leqq r(t^*)x'(t^*) \int_{t^*}^{t} \frac{ds}{r(s)}$$

which yields a contradiction in the limit as $t \to \infty$. Let t_1 be such that $g(t)>T$ for $t \geqq t_1$. It is easy to verify that there is a constant $A \geqq 1$ such that

$$(1) \qquad x(g(t)) \leqq AR_T(g(t)) \qquad \text{for } t \geqq t_1.$$

Multiplying (A) by $\rho(g(t))/f(x(g(t)))\phi(R_T(g(t)))$ and integrating on $[t_1, t]$ we obtain

$$(2) \qquad \begin{aligned} &\frac{\rho(g(t))r(t)x'(t)}{f(x(g(t)))\phi(R_T(g(t)))} + \int_{t_1}^{t} \frac{\rho(g(s))r(s)x'(s)[f(x(g(s)))\phi(R_T(g(s)))]'}{[f(x(g(s)))\phi(R_T(g(s)))]^2} ds \\ &\quad = C + \int_{t_1}^{t} \frac{r(s)x'(s)\rho'(g(s))g'(s)}{f(x(g(s)))\phi(R_T(g(s)))} ds - \int_{t_1}^{t} \frac{\rho(g(s))a(s)}{\phi(R_T(g(s)))} ds, \end{aligned}$$

where C is a constant.

Since x, f, g, ϕ, R_T are nondecreasing, the integral on the left hand side of (2) is nonnegative. Using the inequalities $r(t)x'(t) \leqq r(g(t))x'(g(t))$, $(r(t)\rho'(t))' \leqq 0$ and (1), and applying the well known Bonnet's theorem, the first integral on the right hand side of (2) is estimated as follows:

$$\begin{aligned} \int_{t_1}^{t} \frac{r(s)x'(s)\rho'(g(s))g'(s)}{f(x(g(s)))\phi(R_T(g(s)))} ds &\leqq \int_{t_1}^{t} \frac{r(g(s))x'(g(s))\rho'(g(s))g'(s)}{f(x(g(s)))\phi(R_T(g(s)))} ds \\ &\leqq r(g(t_1))\rho'(g(t_1)) \int_{t_1}^{t} \frac{x'(g(s))g'(s)}{f(x(g(s)))\phi(R_T(g(s)))} ds \\ &\leqq Ar(g(t_1))\rho'(g(t_1)) \int_{x(g(t_1))/A}^{x(g(t_1))/A} \frac{dy}{f(y)\phi(y)}. \end{aligned}$$

Thus the first integral on the right side of (2) remains bounded above as $t \to \infty$. Letting $t \to \infty$ in (2) we conclude that

$$\lim_{t \to \infty} \frac{\rho(g(t))r(t)x'(t)}{f(x(g(t)))\phi(R_T(g(t)))} = -\infty,$$

which contradicts the fact that $x'(t) \geqq 0$ for $t \geqq t_1$. This completes the proof of the theorem.

Remark. Theorem 1 extends a recent result of the authors [3, Theorem 1] for the special case of equation (A) with $r(t) \equiv 1$.

Corollary 1.1 (Bykov, Bykova and Šercov [1]). *Assume that there is* $\varepsilon>0$ *such that*

$$\int^{\infty} [R_T(g(t))]^{1-\epsilon} a(t)dt = \infty \qquad \text{for any } T > 0.$$

Then all solutions of the equation

$$(r(t)x'(t))' + a(t)x(g(t)) = 0$$

are oscillatory.

Proof. Apply Theorem 1 to the particular case where $f(y) = y$, $\rho(t) = R_T(t)$, $\phi(y) = y^{\epsilon}$.

Corollary 1.2 (Bykov, Bykova and Šercov [1]). *Assume that*

$$\int^{\infty} R_T(g(t))a(t)dt = \infty \qquad \text{for any } T > 0.$$

Then all solutions of the equation

$$(r(t)x'(t))' + a(t)\,|x(g(t))|^{\alpha} \operatorname{sgn} x(g(t)) = 0, \qquad \alpha > 1,$$

are oscillatory.

Proof. Apply Theorem 1 to the particular case where $f(y) = |y|^{\alpha} \operatorname{sgn} y$, $\alpha > 1$, $\rho(t) = R_T(t)$, $\phi(y) \equiv 1$.

2. The object of this section is to prove an oscillation theorem for (A) which is particularly useful to the case $\int^{\infty} \frac{dt}{r(t)} < \infty$.

Theorem 2. *Assume there exists a positive function* $\sigma(t) \in C^2(0, \infty)$ *with the properties:*

$$\sigma'(t) \leqq 0, \qquad (r(t)\sigma'(t))' \geqq 0,$$

$$\int^{\infty} \frac{dt}{\sigma(t)r(t)} = \infty,$$

$$\int^{\infty} \sigma(t)a(t)dt = \infty.$$

Let $\int_{\pm 0}^{\pm \delta} \frac{dy}{f(y)} < \infty$ *for some* $\delta > 0$. *Then all solutions of* (A) *are oscillatory.*

Proof. This theorem was motivated by Kamenev [2]. Let $x(t)$ be a nonoscillatory solution such that $x(g(t)) > 0$ for $t \geqq t_1$. It follows that $r(t)x'(t)$ is nonincreasing for $t \geqq t_1$ and so $x'(t)$ is eventually of constant sign. We multiply (A) by $\sigma(t)/f(x(g(t)))$ and integrate from t_1 to t to obtain

$$\frac{\sigma(t)r(t)x'(t)}{f(x(g(t)))} + \int_{t_1}^{t} \frac{\sigma(s)r(s)x'(s)[f(x(g(s)))]'}{[f(x(g(s)))]^2} ds = C + \int_{t_1}^{t} \frac{r(s)x'(s)\sigma'(s)}{f(x(g(s)))} ds - \int_{t_1}^{t} \sigma(s)a(s)ds, \tag{3}$$

where C is a constant. It is clear that the integral on the left side of (3) is nonnegative.

Let $x'(t) \geqq 0$. Then, the first integral on the right side of (3) is nonpositive, and therefore, letting $t \to \infty$ in (3), we get a contradiction.

Let $x'(t) \leqq 0$. Then, as in the proof of Theorem 1, we can show that the first integral on the right side of (3) is bounded above. We can choose $t_2 \geqq t_1$ so that the right hand side of (3) is less than -1, i.e.,

$$(4) \qquad 1+\int_{t_2}^{t} \frac{\sigma(s)r(s)x'(s)[f(x(g(s)))]'}{[f(x(g(s)))]^2}ds \leqq \frac{\sigma(t)r(t)(-x'(t))}{f(x(g(t)))}$$

for $t \geqq t_2$. Multiplying both sides of (4) by

$$-\frac{[f(x(g(t)))]'}{f(x(g(t)))}\left\{1+\int_{t_2}^{t} \frac{\sigma(s)r(s)x'(s)[f(x(g(s)))]'}{[f(x(g(s)))]^2}ds\right\}^{-1} \geqq 0$$

and integrating from t_2 to t, we have

$$(5) \qquad \log\frac{f(x(g(t_2)))}{f(x(g(t)))} \leqq \log\left\{1+\int_{t_2}^{t} \frac{\sigma(s)r(s)x'(s)[f(x(g(s)))]'}{[f(x(g(s)))]^2}ds\right\}.$$

From (4) and (5) we get

$$f(x(g(t_2))) \leqq -\sigma(t)r(t)x'(t),$$

or

$$x(t)-x(t_2) \leqq -f(x(g(t_2)))\int_{t_2}^{t} \frac{ds}{\sigma(s)r(s)},$$

which gives $\lim_{t\to\infty} x(t) = -\infty$, a contradiction. This proves the theorem.

Corollary 2.1. *Consider the equation*

$$(6) \qquad (r(t)x'(t))' + a(t)\,|x(g(t))|^{\alpha}\operatorname{sgn} x(g(t)) = 0, \qquad 0<\alpha<1.$$

Assume that

$$\int^{\infty} \frac{dt}{r(t)} < \infty,$$

$$\int^{\infty} S(t)a(t)dt = \infty,$$

where $S(t)=\int_t^{\infty} \frac{ds}{r(s)}$. *Then all solutions of* (6) *are oscillatory.*

Proof. Apply Theorem 2 to the particular case where $f(y) = |y|^{\alpha}\operatorname{sgn} y$, $0<\alpha<1$, $\sigma(t)=\int_t^{\infty}\frac{ds}{r(s)}$.

3. Let us consider the damped equation (B). Assume that $p(t)$, $q(t) \in C(0,\infty)$, $q(t) \geqq 0$ and $g(t)$ satisfies condition (c).

Theorem 3. *Suppose that* $tp(t) \leqq 1$ *and* $(tp(t))' \geqq 0$ *for sufficiently large* t *and let*

$$\int_{\pm\delta}^{\pm\infty} \frac{dy}{f(y)} < \infty \qquad \text{for some } \delta>0,$$

$$\int^{\infty} g(t)q(t)\exp\left(\int_{g(t)}^{t} p(s)ds\right)dt = \infty.$$

Then all solutions of (B) *are oscillatory.*

Proof. Equation (B) can be transformed into an equation of the form (A) where $r(s)=\exp\left(\int_T^t p(s)ds\right)$ and $a(t)=r(t)q(t)$. If we choose $\rho(t)=t/r(t)$ and $\phi(y)\equiv 1$, then the assumptions of the theorem guarantee that those of Theorem 1 are all satisfied, and the assertion follows from Theorem 1.

Theorem 4. *Assume that $tp(t) \geqq 1$ and $(tp(t))' \leqq 0$ for sufficiently large t and let*

$$\int_{\pm 0}^{\pm \delta} \frac{dy}{f(y)} < \infty \qquad \text{for some } \delta > 0,$$

$$\int^{\infty} tq(t)dt = \infty.$$

Then all solutions of (B) *are oscillatory.*

Proof. Choose $\sigma(t) = t/r(t)$ and apply Theorem 2 to equation (A) into which equation (B) is transformed.

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References

[1] Ya. V. Bykov, L. Ya. Bykova, and E. I. Šercov: Sufficient conditions for oscillation of solutions of nonlinear differential equations with deviating argument. Differencial'nye Uravnenija, **9**, 1555–1560 (1973) (in Russian).

[2] I. V. Kamenev: On the oscillation of solutions of a nonlinear second order equation with coefficient of variable sign. Differencial'nye Uravnenija, **6**, 1718–1721 (1970) (in Russian).

[3] T. Kusano and H. Onose: Oscillations of functional differential equations with retarded argument. J. Differential Equations, **15**, 269–277 (1974).