107. *Initial-Boundary Value Problems of Some Non-Linear Evolution Equations in Orlicz-Sobolev Spaces*

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1. Introduction. Let Ω be a bounded domain in R^n with boundary $\partial\Omega$. Recently in his paper [1] T. Donaldson proved the existence of weak solutions (in some Orlicz-Sobolev spaces) of non-linear elliptic boundary value problems of which are given two examples

$$\sum_{i,j\leqslant n} D_i(u\cdot\exp(D_ju)^2)+D_n(\exp\beta(D_nu)^2)=f,\ \beta>0 \tag{1.1}$$

and

$$\sum_{i,j\leqslant n} D_i(D_ju)^2\ln(D_ju)^2=f \tag{1.2}$$

both associated with the boundary condition $u|_{\partial\Omega}=0$.

Originally Leray and Lions suggest in [4] an introduction of Orlicz-Sobolev spaces for those problems as (1.1), (1.2).

In this paper we consider the initial-boundary value problems for evolution equations of the form

$$\frac{\partial u}{\partial t}+Au=f \tag{1.3}$$

with conditions

$$u(x,0)=u_0(x) \tag{1.4}$$

$$u|_{\partial\Omega}=0 \tag{1.5}$$

in some Orlicz-Sobolev spaces where Au are of a growth not equivalent to any power and are similar to (1.2). Our equations (1.3) furnish a simple example:

$$\frac{\partial u}{\partial t}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)\ln\left(\left|\frac{\partial u}{\partial x_i}\right|+1\right)=f,\qquad p\geqslant 2.$$

2. Preliminaries. In this section we give some necessary definitions and lemmas from Orlicz spaces which are referred to in [3], [2]. We call a function an N-function if it admits of the representation

$$M(\xi)=\int_0^{|\xi|}p(t)dt \tag{2.1}$$

where the function $p(t)$ is upper-continuous for $t\geqslant 0$, positive for $t>0$ and non-decreasing with conditions

$$p(0)=0,\qquad \lim_{t\to\infty}p(t)=\infty.$$

$M(\xi)$, a real-valued function on R^1, is an N-function if and only if $M(\xi)$ is a continuous even function which is convex, increasing for $u\geqslant 0$ and satisfies

$$\lim_{\xi\to 0}\frac{M(\xi)}{\xi}=0, \qquad \lim_{\xi\to\infty}\frac{M(\xi)}{\xi}=\infty.$$

$N(\eta)$, a real-valued function on R^1, is said to be the complementary N-function to $M(\xi)$ if it admits of the representation

$$N(\eta)=\int_0^{|\eta|} q(s)ds, \qquad q(s)=\sup\{t\,|\,p(t)\leqslant s\}. \tag{2.2}$$

We denote $M_1(\xi)\prec M_2(\xi)$ if there exist constants $\xi_0\geqslant 0$, $k>0$ such that

$$M_1(\xi)\leqslant M_2(k\xi) \qquad \text{for } \xi\geqslant\xi_0.$$

$M_1(\xi)$ and $M_2(\xi)$ are said to be equivalent and written $M_1(\xi)\sim M_2(\xi)$ if $M_1(\xi)\prec M_2(\xi)$ and $M_2(\xi)\prec M_1(\xi)$.

We say an N-function $M(\xi)$ satisfies the Δ_2-condition if there exist constants $\xi_0\geqslant 0$ and $k>0$ such that

$$M(2\xi)\leqslant kM(\xi) \qquad \text{for } \xi\geqslant\xi_0.$$

The Orlicz class $L_M(\Omega)=L_M$ is the set of functions $u(x)$ such that

$$\int_\Omega M(u(x))dx<+\infty.$$

The Orlicz space $L_M^*(\Omega)=L_M^*$ is the linear hull of L_M. L_M^* is made a Banach space by the Luxemburg norm

$$\|u\|_M=\inf\left\{k\,;\int_\Omega M\left(\frac{u(x)}{k}\right)dx\leqslant 1\right\}.$$

If $M(\xi)$ satisfies the Δ_2-condition, then $L_M^*=L_M$ and L_M^* is separable. $L_{M_1}^*\subset L_{M_2}^*$ holds if and only if $M_2(\xi)\prec M_1(\xi)$. L_M^* is reflexive if and only if $M(\xi)$ and $N(\eta)$ both satisfy the Δ_2-condition.

If $\int_\Omega M(u(x))dx\leqslant C$ (we say, $u(x)$ is "bounded in the mean"), then we have $\|u\|_M\leqslant C+1$. If $\lim_{n\to\infty}\|u_n-u_0\|_M=0$ for $u_n, u_0\in L_M$, then

$$\lim_{n\to\infty}\int_\Omega M(u_n(x)-u_0(x))dx=0$$

(We call this convergence "convergence in the mean").

Here and afterwards $M(\xi)=\int_0^{|\xi|}p(t)dt$ is the given N-function which satisfies the Δ_2-condition.

Examples. $M_1(\xi)=|\xi|^r\ln(|\xi|+1)$ $(r\geqslant 1)$, $M_2(\xi)=|\xi|^r$ $(r>1)$ are both N-functions satisfying the Δ_2-condition for all ξ.

3. Lemmas and main theorem.

Lemma 3.1.

$$\hat{M}(\xi)=\int_0^{|\xi|}M(t)dt$$

is the N-function and satisfies the Δ_2-condition. Moreover, the N-function $\hat{N}(\eta)$ complementary to $\hat{M}(\xi)$ satisfies the Δ_2-condition for all η and admits of the representation

$$\hat{N}(\eta)=\int_0^{|\eta|}M^{-1}(t)dt.$$

For the proof, see [3; Chap. I, § 4, p. 25].

Now define two functions $\varphi(t), \Phi(\xi)$ on R^1 by

$$\varphi(t)=tp(t), \qquad \Phi(\xi)=\int_0^{|\xi|}\varphi(t)dt.$$

Then we have

Lemma 3.2. *$\Phi(\xi)$ is an N-function equivalent to $\hat{M}(\xi)$ and satisfies the Δ_2-condition. Further, let $\Psi(\eta)$ be the complementary N-function to $\Phi(\xi)$ admitting of the representation*

$$\Psi(\eta)=\int_0^{|\eta|}\psi(s)ds.$$

Then $\Psi(\eta)$ also satisfies the Δ_2-condition.

Proof. The N-function $M(\xi)$ satisfies the Δ_2-condition if and only if there exist constants $\alpha>1$ and $\xi_0>0$ such that, for $\xi\geqslant\xi_0$,

$$1<\frac{\xi p(\xi)}{M(\xi)}<\alpha$$

([3; Chap. I, § 4, p. 24]). Thus follows that

$$M(\xi)<\varphi(\xi)<M(\alpha\xi) \qquad \text{for } \xi\geqslant\xi_0,$$

that is, $\hat{M}(\xi)\sim\Phi(\xi)$. Hence $\hat{N}(\eta)\sim\Psi(\eta)$ also holds. Q.E.D.

Lemma 3.3. *Let $p(t)$ and $q(s)$ be both continuous. If $u(x)\in L_\Phi$, then $M(u(x))\in L_\Psi$.*

Proof. It is obvious that the inequality

$$M(\xi)\leqslant\varphi\left(\frac{\varphi(\xi)}{p(\xi)}\right)$$

holds for $\xi>0$. Hence we have

$$p(\xi)\psi(M(\xi))\leqslant\varphi(\xi).$$

By integrating both sides from 0 to $|\xi|$, we obtain

$$\Psi(M(\xi))\leqslant\Phi(\xi) \qquad \text{for all } \xi,$$

i.e.

$$\int_\Omega \Psi(M(u(x)))\,dx\leq\int_\Omega \Phi(u(x))dx. \qquad \text{Q.E.D.}$$

Next we shall prove Poincaré's inequality for N-functions.

Lemma 3.4. *Let $M(\xi)$ be an N-function. If u is a function in L_M^* with compact support in Ω such that $\partial u/\partial x_i$ (in the sense of distribution) $\in L_M^*$. Then the following inequality holds:*

$$\|u\|_M\leqslant d\left\|\frac{\partial u}{\partial x_i}\right\|_M$$

where d is the diameter of Ω.

Proof. Since $L_M^*\subset L^1$, by using Nikodym's theorem, we have

$$u(x_1,\cdots,x_n)=\int_{x_1'}^{x_1}\frac{\partial u}{\partial x_1}(t,x_2,\cdots,x_n)dt \quad \text{a.e.}$$

and

$$|u(x)|\leqslant\int_{x_1'}^{x_1''}\left|\frac{\partial u}{\partial x_1}\right|dx_1$$

where $x_1'=\inf x_1$, $x_1''=\sup x_1$ for $x=(x_1,\cdots,x_n)\in\operatorname{supp}u$, respectively.

By Jensen's integral inequality ([3; Chap. II, § 8, p. 62])

$$M\Big(\frac{1}{x_1''-x_1'}\,\frac{1}{k_0}u(x)\Big)\leqslant M\Big(\frac{1}{x_1''-x_1'}\int_{x_1'}^{x_1''}\Big|\frac{1}{k_0}\,\frac{\partial u}{\partial x_1}\Big|\,dx_1\Big)$$
$$\leqslant \frac{1}{x_1''-x_1'}\int_{x_1'}^{x_1''}M\Big(\frac{1}{k_0}\,\frac{\partial x}{\partial x_1}\Big)dx_1$$

where $k_0=\|\partial u/\partial x_1\|_M$ (note the last term is finite). Hence

$$\int_\Omega M\Big(\frac{1}{dk_0}u(x)\Big)dx\leqslant\int_\Omega M\Big(\frac{1}{k_0}\,\frac{\partial u}{\partial x_1}\Big)dx\leqslant 1$$

i.e. $\|u\|_M\leqslant dk_0=d\,\|\partial u/\partial x_1\|_M$. Q.E.D.

Now we define the Orlicz-Sobolev space W^mL_Φ by the set of functions u such that

$D^\alpha u$(distributional derivatives) $\in L_\Phi$ for α with $|\alpha|\leqslant m$. Then W^mL_Φ is a Banach space with respect to the norm

$$\|u\|_m=\sum_{|\alpha|\leqslant m}\|D^\alpha u\|_\Phi.$$

Let $W_0^mL_\Phi$ be the closure of $\mathcal{D}$ in W^mL_Φ and let $W^{-m}L_\Psi$ be the dual space of $W_0^mL_\Phi$.

Lemma 3.5. *W^mL_Φ is separable and reflexive.*

Lemma 3.6. *$W^{-m}L_\Psi$ consists of distributions u of the form*

$$u=\sum_{|\alpha|\leqslant m}D^\alpha g_\alpha$$

where $g_\alpha\in L_\Psi$ for α with $|\alpha|\leqslant m$.

For the proofs of Lemma 3.5 and Lemma 3.6 see Lions [5; Chap I].

Main theorem. *Let $M(\xi)$ be given an N-function satisfying the Δ_2-condition and the functions $p(t)$ in (2.1) and $q(s)$ in (2.2) be continuous. And further let be given $u_0(x)\in W_0^mL_\Phi$ and $f(x,t)\in L^2(0,T;L^2)$.*

Then there exists one and only one (weak) solution $u(x,t)$ of the equation

$$\frac{\partial u}{\partial t}+\sum_{|\alpha|=m}(-1)^mD^\alpha(M(D^\alpha u)\operatorname{sgn}D^\alpha u)=f$$

satisfying

$$u\in L^\infty(0,T;W_0^mL_\Phi)$$
$$\partial u/\partial t\in L^2(0,T;L^2)$$
$$u(0)=u_0.$$

4. Proof of main theorem. Put

$$Au=\sum_{|\alpha|=m}(-1)^mD^\alpha(M(D^\alpha u)\operatorname{sgn}D^\alpha u).$$

First we show that A is monotone, hemi-continuous and bounded operator from $W_0^mL_\Phi\to W^{-m}L_\Psi$. Then last assertion follows directly from Hölder's inequality ([3; Chap. II, p. 74, p. 80]), Lemma 3.3 and Lemma 3.5. For the first assertion, since M is even and increasing, we have

$$(M(\xi)\operatorname{sgn}\xi-M(\eta)\operatorname{sgn}\eta)(\xi-\eta)\geqslant 0$$

for any $\xi,\eta\in R^1$. Hence A is monotone. Finally, since

$$|(A(u+\varepsilon v),w)|\leqslant\sum_{|\alpha|=m}\int_\Omega M(|D^\alpha u|+\varepsilon|D^\alpha v|)\,|D^\alpha w|\,dx$$

$$\leqslant \sum_{|\alpha|=m} \int_{\Omega} (\Psi(M(|D^\alpha u|+|D^\alpha v|)+\Phi(D^\alpha w))dx$$
$$\leqslant \sum_{|\alpha|=m} \int_{\Omega} (\Phi(|D^\alpha u|+|D^\alpha v|)+\Phi(D^\alpha w))dx$$

for any u, v and w in $W_0^m L_\Phi$ and any $0<\varepsilon\leqslant 1$, A is hemicontinuous by Lebesgue's convergence theorem.

We shall employ the Galerkin's method. Let $\{w_j\}_{j=1,2,\dots}$ be a complete system of functions in $W_0^m L_\Phi$. We look for approximate solutions $u_\nu(x, t)$ in the form

$$u_\nu(t)=\sum_{j=1}^{\nu} g_{j\nu} w_j$$

where the unknown functions $g_{j\nu}$ are to be determined by the following ordinary differential system

(4.1) $\quad (u_\nu'(t), w_j)+\sum_{|\alpha|=m} (M(D^\alpha u_\nu) \operatorname{sgn} D^\alpha u_\nu, D^\alpha w_j)=(f(t), w_j), \quad 1\leqslant j\leqslant \nu$

with initial condition

$$u_\nu(0)=u_{0\nu}=\sum_{j=1}^{\nu} \alpha_{j\nu} w_j$$

where

$$u_{0\nu}\to u_0 \quad \text{in } W_0^m L_\Phi \text{ strongly as } \nu\to\infty.$$

Then we obtain the following a priori estimates:

(4.2) $$\|u_\nu\|_{L^\infty(0,T;W_0^m L_\Phi)}\leqslant C$$

(4.3) $$\|u_\nu'\|_{L^2(0,T;L^2)}\leqslant C.$$

In fact, multiplying (4.1) by $g_{j\nu}'$ and summing up the resulting equations from $j=1$ to ν imply

$$\|u_\nu'(t)\|_{L^2}^2+\sum_{|\alpha|=m} \int_{\Omega} (\hat{M}(D^\alpha u_\nu(t))'dx=(f(t), u_\nu'(t))$$
$$\leqslant \frac{1}{2}\|f(t)\|_{L^2}^2+\frac{1}{2}\|u_\nu'(t)\|_{L^2}^2.$$

Integrating in t both sides we have

$$\frac{1}{2}\|u_\nu'\|_{L^2(0,T;L^2)}^2+\sum_{|\alpha|=m} \int_{\Omega} \hat{M}(D^\alpha u_\nu(t))dx\leqslant C.$$

Thus a priori estimates (4.2), (4.3) are obtained in virtue of Lemmas 3.2 and 3.4.

Hence there exist a function u and a subsequence $\{u_\mu\}$ of $\{u_\nu\}$ such that

$$u_\mu\to u \text{ in } L^\infty(0, T; W_0^m L_\Phi) \quad \text{weakly star,}$$
$$u_\mu'\to u' \text{ in } L^2(0, T; L^2) \quad \text{weakly,}$$
$$u_\mu(T)\to u(T) \text{ in } W_0^m L_\Phi \quad \text{weakly}$$

and

$$Au_\mu\to\chi \text{ in } L^\infty(0, T; W^{-m} L_\Psi) \quad \text{weakly star.}$$

Hemi-continuity and monotonicity of A yield $\chi=Au$ ([6; Chap. II, p. 160]) which implies the function u is a desired solution.

The uniqueness part follows from the monotonycity of A, as usual.

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