106. *The Whitehead Theorems in Shape Theory*

By Kiiti MORITA

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§ 1. **Introduction.** Recently the notion of shape which was originally introduced by K. Borsuk [2] for compact metric spaces has been extended to the case of topological spaces by S. Mardešić [3]. In this paper we shall use the notion of shape in the sense of Mardešić [3]. As in our previous paper [6], let $\pi_n(X, A, x_0)$ be the n-th (Čech) homotopy pro-group of a pair (X, A, x_0) of pointed topological spaces and $H_n(X, A)$ the n-th (Čech) homology pro-group of a pair (X, A) of topological spaces. For a continuous map $f: (X, A, x_0) \to (Y, B, y_0)$ let us denote by $\pi_k(f)$ or $H_k(f)$ the induced morphism of the k-th homotopy or homology pro-groups.

In this paper we shall establish the following theorems as analogues of the classical Whitehead theorems.

Theorem 1. *Let (X, x_0) and (Y, y_0) be connected pointed spaces and let $f: (X, x_0) \to (Y, y_0)$ be a continuous map. For $n \geqq 2$ let us consider the following conditions.*

(i) $\pi_k(f): \pi_k(X, x_0) \to \pi_k(Y, y_0)$

is an isomorphism for $1 \leqq k < n$ and an epimorphism for $k = n$.

(ii) $H_k(f): H_k(X, x_0) \to H_k(Y, y_0)$

is an isomorphism for $1 \leqq k < n$ and an epimorphism for $k = n$.

Then (i) *implies* (ii), *and conversely, if $\pi_1(X, x_0) = 0$ and $\pi_1(Y, y_0) = 0$,* (ii) *implies* (i).

Theorem 2. *Let (X, x_0) and (Y, y_0) be connected pointed spaces of finite dimension and let $n_0 = \max(1 + \dim X, \dim Y)$. If $f: (X, x_0) \to (Y, y_0)$ is a continuous map such that the induced morphism $\pi_k(f)$: $\pi_k(X, x_0) \to \pi_k(Y, y_0)$ is a bimorphism for $1 \leqq k < n_0$ and an epimorphism for $k = n_0$, then f induces a shape equivalence.*

In [4] Mardešić deduced the conclusion of Theorem 2 under a condition that $\pi_k(f)$ is a bimorphism for $1 \leqq k \leqq n_0$ and an epimorphism for $k = n_0 + 1$. For the case of compact metric spaces the same result as in Mardešić [4] was obtained earlier by M. Moszyńska [9].

§ 2. **Preliminaries.** Let $f: (X, x_0) \to (Y, y_0)$ be a continuous map of pointed topological spaces. Let (Z, x_0) be the mapping cylinder of f which is obtained from the topological sum $(X \times I) \cup Y$ (where I is the closed unit interval $[0, 1]$ in the real line) by identifying $(x, 1)$ with $f(x)$ for $x \in X$ and by shrinking $(x_0 \times I) \cup \{y_0\}$ to a point which we denote by the letter x_0. The image of (x, t) and y under this identification will be denoted by $[x, t]$ and $[y]$ respectively. Then there are embeddings $i: (X, x_0) \to (Z, x_0)$ and $j: (Y, y_0) \to (Z, x_0)$ with $i(x) = [x, 0]$ and $j(y) = [y]$. (X, x_0) and (Y, y_0) are regarded as subspaces of Z by these embeddings. Moreover, there is a retraction $r: (Z, x_0) \to (Y, y_0)$ defined by $r[x, t] = [f(x)]$ for $x \in X$, $t \in I$ and $r[y] = [y]$ for $y \in Y$. Then i is a cofibration and $f \simeq ri$, $1_Z \simeq jr$ rel Y, $1_Y = rj$. Thus, r is a homotopy equivalence. Hence f induces a shape equivalence iff i induces a shape equivalence. Also $\pi_k(f): \pi_k(X, x_0) \to \pi_k(Y, y_0)$ is an epimorphism (resp. monomorphism, isomorphism) iff $\pi_k(i): \pi_k(X, x_0) \to \pi_k(Z, x_0)$ is so, and the same holds for morphisms of homology pro-groups (cf. [5, Theorem 2.4]).

Here we note that X is P-embedded in Z. A subspace A of a topological space R is said to be P-embedded in R if every locally finite open normal cover of A has a refinement which can be extended to a locally finite normal open cover of R. Since i is a cofibration, (Z, X) has the homotopy extension property with respect to any topological space and hence by [8, Theorem 3.7] X is P-embedded in Z. Of course, a direct proof can be obtained easily.

In case A is P-embedded in a topological space R, the homotopy pro-group sequence of (R, A, x_0), where $x_0 \in A$, and the homology pro-group sequence of (R, A) are exact.

§3. Proof of Theorem 1. In a previous paper [6] we have proved the following theorem.

Theorem 3. *Let (X, A, x_0) be a pair of pointed connected topological spaces. Suppose that $\pi_k(X, A, x_0) = 0$ for $1 \leqq k \leqq n$. Then $H_k(X, A) = 0$ for $1 \leqq k \leqq n$. Furthermore, if, in addition, $\pi_1(A, x_0) = 0$ and A is P-embedded in X, then the Hurewicz morphism $\Phi_{n+1}(X, A, x_0): \pi_{n+1}(X, A, x_0) \to H_{n+1}(X, A)$ is an isomorphism.*

Now, assume (i) in Theorem 1. Let Z be the mapping cylinder of f described in §2. Then, as is proved by Moszyńska [9, §2, 1.3] (cf. also Mardešić [4, 5.3]), $\pi_k(Z, X, x_0) = 0$ for $1 \leqq k \leqq n$. Since Z and X are connected, Theorem 3 above shows that $H_k(Z, X) = 0$ for $1 \leqq k \leqq n$. Since the homology pro-group sequence of (Z, X) is exact in the pro-category of abelian groups and this pro-category is an abelian category (cf.[1]), we conclude that $H_k(i): H_k(X) \to H_k(Z)$ is an isomorphism for $1 \leqq k < n$ and an epimorphism for $k = n$ (cf. [10, p. 124]).

Conversely, assume (ii) and suppose that $\pi_1(X, x_0) = 0$ and $\pi_1(Y, y_0) = 0$. Then $\pi_1(Z, X, x_0) = 0$. Hence by Theorem 3 $\pi_2(Z, X, x_0)$ is isomorphic to $H_2(Z, X)$. On the other hand, it follows from (ii) and the exactness of the homology pro-group sequence of (Z, X) that $H_k(Z, X) = 0$ for $1 \leqq k \leqq n$. Hence $\pi_2(Z, X, x_0) = 0$. By repeated application of Theorem 3 we have $\pi_k(Z, X, x_0) = 0$ for $3 \leqq k \leqq n$.

As is easily seen, if G and H are pro-abelian groups (= objects in the pro-category of abelian groups) and $f: G \to H$ is a morphism in the pro-category of abelian groups, then the kernel of f in the pro-category of groups coincides with the kernel of f in the pro-category of abelian groups. Hence, a sequence of morphisms in the pro-category of abelian groups is exact if it is exact in the pro-category of groups. Hence for $2 \leqq k < n$, the sequence $0 \to \pi_k(X, x_0) \to \pi_k(Z, x_0) \to 0$ is exact in the pro-category of abelian groups. Therefore $\pi_k(i): \pi_k(X, x_0) \to \pi_k(Z, x_0)$ is an isomorphism in the pro-category of abelian groups and hence in the pro-category of groups.

On the other hand, since $\pi_n(X, x_0) \to \pi_n(Z, x_0) \to 0$ is exact, $\pi_n(i)$: $\pi_n(X, x_0) \to \pi_n(Z, x_0)$ is an epimorphism in the pro-category of groups. Thus, the proof of Theorem 1 is completed.

§ 4. **Proof of Theorem 2.** Let Z be the mapping cylinder of a continuous map $f: (X, x_0) \to (Y, y_0)$ which is described in § 2. Suppose that $n_0 = \max(1 + \dim X, \dim Y) < \infty$. Here the covering dimension of a space R, $\dim R$ in notation, is defined to be the least integer n such that every locally finite normal open cover of R is refined by a locally finite normal open cover of R of order $\leqq n+1$. As is shown by Mardešić [4], we have $\dim Z \leqq n_0$. Since $\pi_k(i): \pi_k(X, x_0) \to \pi_k(Z, x_0)$ is a bimorphism for $1 \leqq k < n_0$ and an epimorphism for $k = n_0$, we have $\pi_k(Z, X, x_0) = 0$ for $1 \leqq k \leqq n_0$. Therefore, Theorem 2 follows from Theorem 4 below.

Theorem 4. *Let (X, A, x_0) be a pair of pointed connected topological spaces such that A is P-embedded in X. If* $\dim X/A \leqq n$ *and* $\pi_k(X, A, x_0) = 0$ *for* $1 \leqq k \leqq n$, *then the inclusion map* $i: (A, x_0) \to (X, x_0)$ *induces a shape equivalence.*

Proof of Theorem 4. Since $\dim X/A \leqq n$, by Morita [7, Theorem 3] there is an inverse system $\{(X_\lambda, A_\lambda, x_{0\lambda}), [p_{\lambda\lambda'}], \Lambda\}$ in the homotopy category of pairs of pointed simplicial complexes with the weak topology such that it is isomorphic to the Čech system of (X, A, x_0) as defined in [6] and $\dim X_\lambda / A_\lambda \leqq n$ for each $\lambda \in \Lambda$.

Since $\pi_k(X, A, x_0) = 0$ for $1 \leqq k \leqq n$, for each $\lambda \in \Lambda$ there is $\mu \in \Lambda$ which admits a sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_n\}$ in Λ such that $\lambda < \lambda_0 < \cdots < \lambda_n < \mu$ and

$$\pi_k(p_{\lambda_j \lambda_{j+1}}): \pi_k(X_{\lambda_{j+1}}, A_{\lambda_{j+1}}, x_{0\lambda_{j+1}}) \to \pi_k(X_{\lambda_j}, A_{\lambda_j}, x_{0\lambda_j})$$

is a zero homomorphism for $1 \leqq k \leqq n$ and $0 \leqq j \leqq n-1$. In such a case we write $\lambda \prec \mu$. In case $\lambda \prec \mu$, since $\dim X_\mu / A_\mu \leqq n$, [6, Lemma 3] there is a continuous map $\psi_{\lambda\mu}: (X_\mu, x_{0\mu}) \to (A_\lambda, x_{0\lambda})$ such that

$$i_\lambda \psi_{\lambda\mu} \simeq p_{\lambda\mu} | (X_\mu, x_{0\mu}): (X_\mu, x_{0\mu}) \to (X_\lambda, x_{0\lambda}),$$
$$\psi_{\lambda\mu} i_\mu \simeq p_{\lambda\mu} | (A_\mu, x_{0\mu}): (A_\mu, x_{0\mu}) \to (A_\lambda, x_{0\lambda}),$$

where $i_\lambda: (A_\lambda, x_{0\lambda}) \to (X_\lambda, x_{0\lambda})$ is the inclusion map. Then by [6, Lemma

1] $(\Lambda, \prec)$ is a directed set and $\{(X_\lambda, A_\lambda, x_{0\lambda}), [p_{\lambda\lambda'}], (\Lambda, \prec)\}$ is isomorphic to the original inverse system $\{(X_\lambda, A_\lambda, x_{0\lambda}), [p_{\lambda\lambda'}], (\Lambda, \prec)\}$. By virtue of [6, Lemma 2] the existence of $\psi_{\lambda\mu}$ for $\lambda, \mu \in \Lambda$ with $\lambda \prec \mu$ shows that the inclusion map $i: (A, x_0) \to (X, x_0)$ induces an isomorphism from $\{(A_\lambda, x_{0\lambda}), [p_{\lambda\lambda'} | (A_{\lambda'}, x_{0\lambda'})], (\Lambda, \prec)\}$ to $\{(X_\lambda, x_{0\lambda}), [p_{\lambda\lambda'} | (X_{\lambda'}, x_{0\lambda'})], (\Lambda, \prec)\}$ in the pro-category of the homotopy category of pointed CW complexes.

On the other hand, in a previous paper [5] we have proved that the Čech construction, which assigns to any pointed space (X, x_0) the Čech system of (X, x_0) as defined in [6], yields a category-equivalence from the shape category of pointed topological spaces to a full subcategory of the pro-category of the homotopy category of pointed CW complexes. Therefore, $i: (A, x_0) \to (X, x_0)$ induces a shape equivalence. This completes the proof of Theorem 4.

Added in Proof. Recently the author has proved that Theorems 1 and 2 remain true even in case f is a shape morphism, and that if f is a shape morphism such that $\pi_k(f)$ is an isomorphism for $1 \leqq k \leqq \max(\dim X, \dim Y) < \infty$, then f is a shape equivalence.

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