96. *Fourier Transform of Banach Algebra Valued Functions on Group. II**)

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The next theorem is a generalization of the theorem in the previous paper.

Theorem. *Let h be a continuous mapping of $L^1(G\to A)$ into B with the following properties;*

(1) $h(af+bg)=ah(f)+bh(g)$ *for any complex numbers a, b, and f, $g\in L^1(G\to A)$,*

(2) $h(f*g)=h(f)\cdot h(g)$ *for $f,g\in L^1(G\to A)$,*

(3) *for any $\varepsilon>0$ there exists $f_\varepsilon\in L^1(G\to A)$ such that $\|h(f_\varepsilon)-1\|_B<\varepsilon$.*

Then there exist a homomorphism α of A into B and a bounded continuous homomorphism φ of G into $C_B(\alpha(A))$ such that

$$h(f)=\int_G \varphi(x)\alpha(f(x))dx, \qquad for\ f\in L^1(G\to A),$$

where $C_B(\alpha(A))$ means the set of all elements of B that commute with every element in the range of α.

Proof. By the property (3), there exists $f_1\in L^1(G\to A)$ such that $h(f_1)^{-1}$ exists in B. For this f_1 and for any fixed $f\in L^1(G\to A)$, by Proposition 4, there exists a sequence $\{E_n\}$ of measurable sets in G such that

$$|||m(E_n)^{-1}\chi_{E_n}*f_1-f_1|||<1/n,$$
$$|||m(E_n)^{-1}\chi_{E_n}*f-f|||<1/n, \qquad (n=1,2,\cdots).$$

Then, for $a\in A$,

$$||m(E_n)^{-1}h(\chi_{E_n}*af_1)-h(af_1)||_B=||m(E_n)^{-1}h(a\chi_{E_n})h(f_1)-h(af_1)||_B \leqq ||h||\cdot||a||/n,$$

which vanishes as n tends to ∞.

We put $\alpha(a)=\lim_{n\to\infty} m(E_n)^{-1}h(a\chi_{E_n})=h(af_1)h(f_1)^{-1}$. Replacing f_1 by f in the inequality above, we get $h(af)=\alpha(a)h(f)$.

Since the definition of α does not depend on the choice of $\{E_n\}$, $h(af)=\alpha(a)h(f)$ holds good for every $f\in L^1(G\to A)$.

We show α is a homomorphism.

$$\alpha(ab)=\alpha(ab)h(f_1)h(f_1)^{-1}=h(abf_1)h(f_1)^{-1}=\alpha(a)\alpha(b)h(f_1)h(f_1)^{-1} =\alpha(a)\alpha(b).$$

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*) Continuation of the same titled paper, published in this Proceedings, June 1974.

Now let φ be the bounded continuous homomorphism of G into B which is constructed in Proposition 5. Then φ has following properties;

(i) $h(f_t)=\varphi(t)h(f)$ for $t \in G$ and $f \in L^1(G\to A)$,

(ii) $\varphi(st)=\varphi(s)\varphi(t)$ for $s, t \in G$,

(iii) $\alpha(a)\varphi(t)=\varphi(t)\alpha(a)$ for all $a \in A$ and $t \in G$.

We have (iii) because of

$$\alpha(a)\varphi(t)h(f_1)=h(af_{1t})=h((af_1)_t)=\varphi(t)\alpha(a)h(f_1).$$

If f is a measurable step function, $f=\sum_{\nu=1}^{n} a_\nu \chi_{E_\nu}$, then we have

$$\alpha(f(x))=\alpha\left(\sum_{\nu=1}^{n} a_\nu \chi_{E_\nu}(x)\right)=\sum_{\nu=1}^{n} \alpha(a_\nu \chi_{E_\nu}(x))=\sum_{\nu=1}^{n} \alpha(a_\nu)\chi_{E_\nu}(x),$$

and

$$\begin{aligned} h(f)&=\sum_{\nu=1}^{n} h(a_\nu \chi_{E_\nu})=\sum_{\nu=1}^{n} h(a_\nu \chi_{E_\nu} * f_1)h(f_1)^{-1} \\ &=\sum_{\nu=1}^{n} h(\chi_{E_\nu})h(a_\nu f_1)h(f_1)^{-1} \\ &=\sum_{\nu=1}^{n} h(\chi_{E_\nu})\alpha(a_\nu) \\ &=\sum_{\nu=1}^{n} \int_G \varphi(x)\chi_{E_\nu}(x)\alpha(a_\nu)dx \\ &=\int_G \varphi(x) \sum_{\nu=1}^{n} \alpha(a_\nu)\chi_{E_\nu}(x)dx \\ &=\int_G \varphi(x)\alpha(f(x))dx. \end{aligned}$$

If we choose any $g \in L^1(G\to A)$, then we can also choose a measurable step function $f=\sum_{\nu=1}^{n} a_\nu \chi_{E_\nu}$ such that $|||g-f|||<\varepsilon/2 \max(\|h\|, \|\varphi\|_\infty)$.

Hence we get a following inequality.

$$\begin{aligned} &\left\| h(g)-\int_G \varphi(x)\alpha(g(x))dx \right\|_B \\ &\leqq \|h(g)-h(f)\|_B+\left\|\int_G \varphi(x)\alpha(f(x))dx-\int_G \varphi(x)\alpha(g(x))dx\right\|_B \\ &\leqq \|h\|\cdot|||g-f|||+\int_G \|\varphi(x)\|_B\cdot\|\alpha(f(x)-g(x))\|_B dx \\ &<\varepsilon/2+\varepsilon/2=\varepsilon. \end{aligned}$$

Q.E.D.