167. *Normal Expectations and Crossed Products of von Neumann Algebras*

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In this paper, we shall show that a von Neumann algebra $\mathcal{M}$ is isomorphic to the crossed product $G\otimes\mathcal{A}$ of a von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ by a group G of automorphisms of $\mathcal{A}$ implemented by a unitary group in $\mathcal{M}$ under certain conditions. This result is a generalization of two theorems of Golodets [3].

1. Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$, and let G be a discrete group of (*-) automorphisms of $\mathcal{A}$.

On the Hilbert space $\mathfrak{H}\otimes l^2(G)$, the tensor product of $\mathfrak{H}$ and $l^2(G)$, define a representation I of $\mathcal{A}$ by

$$I(A)\Big(\sum_{g\in G}\xi_g\otimes\varepsilon_g\Big)=\sum_{g\in G}g^{-1}(A)\xi_g\otimes\varepsilon_g,$$

for each A in $\mathcal{A}$ and ξ_g in $\mathfrak{H}$, where ε_g is an orthonormal basis in $l^2(G)$ such that

$$\varepsilon_g(h)=\begin{cases}1 & (g=h)\\ 0 & (g\neq h),\end{cases}\qquad g,h\in G.$$

Letting G act as a permutation group in $\mathfrak{H}\otimes l^2(G)$, we obtain a unitary representation V_g of G such that

$$V_g\Big(\sum_{h\in G}\xi_h\otimes\varepsilon_h\Big)=\sum_{h\in G}\xi_{g^{-1}h}\otimes\varepsilon_h,\qquad g\in G,\xi_h\in\mathfrak{H}.$$

One can then verify that I is a faithful normal representation with the covariance formula

$$V_gI(A)V_g^*=I(g(A)),\qquad g\in G, A\in\mathcal{A}.$$

Then the von Neumann algebra acting on $\mathfrak{H}\otimes l^2(G)$ generated by $I(\mathcal{A})$ and V_G is called the *crossed product* $G\otimes\mathcal{A}$ of $\mathcal{A}$ by G.

Theorem 1. *Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$, $\mathcal{A}$ a von Neumann subalgebra of $\mathcal{M}$ and G a discrete group of automorphisms of $\mathcal{A}$. Assume that $(\mathcal{M},\mathcal{A},G)$ satisfies the following three conditions;*

(1) *there is a unitary representation U_g of G into $\mathcal{M}$ with $g(A)=U_gAU_g^*$ for g in G and A in $\mathcal{A}$,*

(2) *$\mathcal{M}$ admits a cyclic vector ξ with $(U_gA\xi,\xi)=0$ for $g(\neq 1)$ in G and A in $\mathcal{A}$,*

and

(3) *$\mathcal{M}$ is generated by $\mathcal{A}$ and U_G.*

Then $\mathcal{M}$ is spatially isomorphic to the crossed product $G\otimes\mathcal{A}$ of $\mathcal{A}$ by G on $\mathfrak{H}_0\otimes l^2(G)$ for some subspace $\mathfrak{H}_0$ of $\mathfrak{H}$.

For the proof of Theorem 1, we need the following lemma.

Lemma 2. *Under the same assumption as in Theorem 1, the Hilbert space $\mathfrak{H}$ equals to the direct sum $\sum_{g\in G}\oplus U_g\mathfrak{H}_0$, where $\mathfrak{H}_0$ is the subspace of $\mathfrak{H}$ generated by the set $\{A\xi\,;\,A\in\mathcal{A}\}$.*

Proof. By the assumptions (1) and (2),

$$(U_gB\xi, A\xi)=(U_g g^{-1}(A^*)B\xi,\xi)=0, \qquad 1\neq g\in G, A, B\in\mathcal{A}.$$

It implies that $\mathfrak{H}_0$ is orthogonal to $U_g\mathfrak{H}_0$ for every $g(\neq 1)$ in G. Let x be an element in $\mathfrak{H}$ which is orthogonal to $U_g\mathfrak{H}_0$ for every g in G. Since $\mathcal{M}$ is generated by $\mathcal{A}$ and U_G, then we have, by the definition of $\mathfrak{H}_0$, that $(x, T\xi)=0$ for every T in $\mathcal{M}$. On the other hand, ξ is a cyclic vector for $\mathcal{M}$. Hence we have that $x=0$. Thus, $\mathfrak{H}$ is represented to the direct sum of $\{U_g\mathfrak{H}_0\,;\, g\in G\}$.

Now, we shall prove Theorem 1.

By Lemma 2, every element x in $\mathfrak{H}$ is represented by the form

$$x=\sum_{g\in G}U_gx_g, \qquad x_g\in\mathfrak{H}_0.$$

Define a mapping W of $\mathfrak{H}$ to $\mathfrak{H}_0\otimes l^2(G)$ by

$$W\Big(\sum_{g\in G}U_gx_g\Big)=\sum_{g\in G}x_g\otimes\varepsilon_g, \qquad x_g\in\mathfrak{H}_0,$$

then W is an isomorphism of $\mathfrak{H}$ onto $\mathfrak{H}_0\otimes l^2(G)$.

By the definition of $\mathfrak{H}_0$, we can regard $\mathcal{A}$ as a von Neumann algebra acting on $\mathfrak{H}_0$. Then the crossed product $G\otimes\mathcal{A}$ of $\mathcal{A}$ by G is the von Neumann algebra on the Hilbert space $\mathfrak{H}_0\otimes l^2(G)$.

Let Φ be a mapping of $\mathcal{M}$ to $\mathcal{L}\,(\mathfrak{H}_0\otimes l^2(G))$ by

$$\Phi(T)=WTW^*$$

for T in $\mathcal{M}$, then Φ satisfies that

$$\Phi(A)=I(A), \qquad A\in\mathcal{A},$$

and that

$$\Phi(U_g)=V_g, \qquad g\in G.$$

In fact,

$$\begin{aligned}\Phi(A)\Big(\sum_{g\in G}x_g\otimes\varepsilon_g\Big)&=WAW^*\Big(\sum_{g\in G}x_g\otimes\varepsilon_g\Big)=WA\Big(\sum_{g\in G}U_gx_g\Big)\\&=W\sum_{g\in G}(U_gg^{-1}(A)x_g)=\sum_{g\in G}g^{-1}(A)x_g\otimes\varepsilon_g\\&=I(A)\Big(\sum_{g\in G}x_g\otimes\varepsilon_g\Big),\end{aligned}$$

for all A in $\mathcal{A}$ and x_g in $\mathfrak{H}_0$, and

$$\begin{aligned}\Phi(U_g)\Big(\sum_{h\in G}x_h\otimes\varepsilon_h\Big)&=WU_g\Big(\sum_{h\in G}U_hx_h\Big)=W\Big(\sum_{h\in G}U_{gh}x_h\Big)\\&=W\Big(\sum_{h\in G}U_hx_{g^{-1}h}\Big)=\sum_{h\in G}x_{g^{-1}h}\otimes\varepsilon_h\\&=V_g\Big(\sum_{h\in G}x_h\otimes\varepsilon_h\Big),\end{aligned}$$

for all g in G and x_h in $\mathfrak{H}_0$.

On the other hand, $\mathcal{M}$ is generated by $\mathcal{A}$ and U_G, and $G\otimes\mathcal{A}$ is generated by $I(\mathcal{A})$ and V_G.

Thus Φ is an spatial isomorphism of $\mathcal{M}$ onto $G\otimes\mathcal{A}$.

Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{A}$ a von Neumann subalgebra of $\mathcal{M}$. A positive linear mapping e of $\mathcal{M}$ onto $\mathcal{A}$ is called an *expectation* of $\mathcal{M}$ onto $\mathcal{A}$ if e satisfies that $e(I)=I$ and $e(AB)=Ae(B)$ for all A in $\mathcal{A}$ and B in $\mathcal{M}$. An expectation e of $\mathcal{M}$ onto $\mathcal{A}$ is called *faithful* if $e(T)=0$ for a positive operator T in $\mathcal{M}$ implies that $T=0$. An expectation e of $\mathcal{M}$ onto $\mathcal{A}$ is called *normal* if $A_\alpha \uparrow A$ implies $e(A_\alpha)\uparrow e(A)$ for A_α and A in $\mathcal{M}$.

As a corollary of Lemma 2, we shall obtain the following:

Corollary 3. *Under the same assumption as in Theorem* 1, *there exists a faithful normal expectation e of $\mathcal{M}$ onto $\mathcal{A}$ such that $e(U_g)=0$ for all $g(\neq 1)$ in G.*

Proof. Let E be the projection of $\mathfrak{H}$ on $\mathfrak{H}_0$ (the subspace in Lemma 2), then E is contained in the commutant $\mathcal{A}'$ of $\mathcal{A}$, and $\{g(E)\,;\,g\in G\}$ is an orthogonal family of projections in $\mathcal{A}'$ with $\sum_{g\in G} g(E)=1$ because $\mathfrak{H}$ is the direct sum of $\{U_g\mathfrak{H}_0\,;\,g\in G\}$ by Lemma 2. That is, G is a shift of $\mathcal{A}'$ in the sense of [2] if we define an action of g in G by

$$g(A')=U_gA'U_g^*$$

for every A' in $\mathcal{A}'$.

Put

$$e(T)=\sum_{g\in G} g(E)Tg(E)$$

for every T in $\mathcal{M}$, then it is clear that $e(U_g)=0$ if $g\neq 1$ and that $e(A)=A$ for all A in $\mathcal{A}$. Since $\mathcal{M}$ is generated by $\mathcal{A}$ and U_G, it follows that e is a mapping of $\mathcal{M}$ onto $\mathcal{A}$. So, e is a normal expectation of $\mathcal{M}$ onto $\mathcal{A}$. Take T in $\mathcal{M}$ such that $e(T^*T)=0$. By the property of $\{g(E)\,;\,g\in G\}$, it follows that $g(E)T^*Tg(E)=0$ for every g in G, so that $Tg(E)=0$, for every g in G. It implies that $T=0$, [1].

Thus e is a faithful normal expectation of $\mathcal{M}$ onto $\mathcal{A}$ with $e(U_g)=0$ for $g\neq 1$.

The following theorem shows that the existence of such an expectation e is also a sufficient condition that $\mathcal{M}$ is isomorphic to $G\otimes\mathcal{A}$.

Theorem 4. *Let $\mathcal{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$, $\mathcal{A}$ a von Neumann subalgebra of $\mathcal{M}$ and G a discrete group of automorphisms of $\mathcal{A}$. Assume that $(\mathcal{M}, \mathcal{A}, G)$ satisfies the following three conditions;*

(1) *there exists a unitary representation U_g of G into $\mathcal{M}$ with $g(A)=U_gAU_g^*$ for g in G and A in $\mathcal{A}$,*

(2′) *there exists a faithful normal expectation e of $\mathcal{M}$ onto $\mathcal{A}$ with*

$e(U_g)=0$ if $g\neq 1$, and

(3) *$\mathcal{M}$ is generated by $\mathcal{A}$ and U_G.*

Then $\mathcal{M}$ is isomorphic to the crossed product $G\otimes\mathcal{A}$ of $\mathcal{A}$ by G.

Proof. Let ψ be a faithful normal state on $\mathcal{A}$. Put

$$\phi(T)=\psi(e(T))$$

for all T in $\mathcal{M}$, then ϕ is a faithful normal state on $\mathcal{M}$.

Let $\mathfrak{R}$ be the representation space of $\mathcal{M}$ by ϕ and $\mathcal{L}(\mathfrak{R})$ the algebra of all bounded operators on $\mathfrak{R}$. Denote by π the isomorphism of $\mathcal{M}$ into $\mathcal{L}(\mathfrak{R})$ defined by ϕ, and by ξ the cyclic vector for $\pi(\mathcal{M})$ in $\mathfrak{R}$. Then (π, ξ) satisfies that $\phi(T)=(\pi(T)\xi, \xi)$ for every T in $\mathcal{M}$.

Letting G act as an automorphism group of $\pi(\mathcal{A})$ such that

$$g(\pi(A))=\pi(U_g)\pi(A)\pi(U_g)^*, \qquad A\in\mathcal{A} \quad \text{and} \quad g\in G,$$

we obtain a unitary representation $\pi(U_g)$ of G into $\pi(\mathcal{M})$ which satisfies the condition (1) in Theorem 1.

By the property of e, we have that

$$(\pi(U_g)\pi(A)\xi, \xi)=\phi(U_gA)=\psi(e(U_gA))=\psi(e(U_g)A)=0,$$

for all A in $\mathcal{A}$ and $g(\neq 1)$ in G. Hence the condition (2) in Theorem 1 is satisfied. It is clear that $\pi(\mathcal{M})$ is generated by $\pi(\mathcal{A})$ and $\pi(U_G)$. Thus by Theorem 1 $\pi(\mathcal{M})$ is isomorphic to the crossed product $G\otimes\pi(\mathcal{A})$ of $\pi(\mathcal{A})$ by G.

On the other hand, by the definition of the action of G on $\pi(\mathcal{A})$, we have that $g(\pi(A))=\pi(g(A))$ for every g in G and A in $\mathcal{A}$. Hence $G\otimes\pi(\mathcal{A})$ is isomorphic to $G\otimes\mathcal{A}$, (see for instance [7, Proposition 3.4]). Therefore $\mathcal{M}$ is isomorphic to $G\otimes\mathcal{A}$.

Let $\mathcal{A}$ be a von Neumann algebra and g an automorphism of $\mathcal{A}$. g is called *freely acting* on $\mathcal{A}$ if for B in $\mathcal{A}$

$$BA=g(A)B, \qquad \text{for all } A \text{ in } \mathcal{A}$$

implies $B=0$. A group G of automorphisms of $\mathcal{A}$ is called *freely acting* on $\mathcal{A}$ if every $g(\neq 1)$ in G is freely acting on $\mathcal{A}$, [4].

Corollary 5. *Let $\mathcal{M}$ be a von Neumann algebra, $\mathcal{A}$ a von Neumann subalgebra of $\mathcal{M}$ and G a discrete group of automorphisms of $\mathcal{A}$. Assume that $(\mathcal{M}, \mathcal{A}, G)$ satisfies the following three conditions;*

(1) *there exists a unitary representation U_g of G into $\mathcal{M}$ with $g(A)=U_gAU_g^*$ for g in G and A in $\mathcal{A}$,*

(2′) *there exists a faithful normal expectation e of $\mathcal{M}$ onto $\mathcal{A}$ and G is freely acting on $\mathcal{A}$, and*

(3) *$\mathcal{M}$ is generated by $\mathcal{A}$ and U_G.*

Then $\mathcal{M}$ is isomorphic to the crossed product $G\otimes\mathcal{A}$ of $\mathcal{A}$ by G.

Proof. Take a $g(\neq 1)$ in G. By the assumption (1),

$$U_gA=g(A)U_g$$

for all A in $\mathcal{A}$. It implies that

$$e(U_g)A=g(A)e(U_g)$$

for all A in $\mathcal{A}$. Since g is freely acting on $\mathcal{A}$, it follows that $e(U_g)=0$.

Thus $(\mathcal{M}, \mathcal{A}, G)$ satisfies the conditions (1), (2′) and (3) in Theorem 4. Hence, by Theorem 4, $\mathcal{M}$ is isomorphic to $G \otimes \mathcal{A}$.

This corollary is a generalization of [3, Theorem 1.5.2] and [3, Theorem 1.7.2].

Theorem 1.5.2 in [3] is that for $(\mathcal{M}, \mathcal{A}, G)$ with the properties (1) and (3) in Corollary 5, where $\mathcal{M}$ is a II_1-factor, $\mathcal{A}$ is a II_1-subfactor of $\mathcal{M}$ and G is an outer automorphism group of $\mathcal{A}$. Theorem 1.7.2 in [3] is that for $(\mathcal{M}, \mathcal{A}, G)$ with the properties (1) and (3) in Corollary 5, where $\mathcal{M}$ is a von Neumann algebra with a finite trace, $\mathcal{A}$ is a von Neumann subalgebra of $\mathcal{M}$ and G is an automorphism group of $\mathcal{A}$ freely acting on the center of $\mathcal{A}$.

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