191. A Note on the Normal Subgroups of 4-fold Transitive Permutation Groups of Degree $5m + 4$

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1. Introduction. The following is the well-known theorem of Jordan ([8], p. 65).

Theorem (Jordan). Let $G$ be a $k$-transitive permutation group on a set $\Omega$, $k \geq 2$, and $G$ be not the symmetric group on $\Omega$. If $H \neq 1$ is a normal subgroup of $G$, then $H$ is $(k-1)$-transitive on $\Omega$ with the exception when $|\Omega|$ is a power of 2 and $k = 3$, and in this exceptional case $H$ may be an elementary abelian 2-group transitive and regular on $\Omega$.

This theorem is refined by A. Wagner [7], N. Ito [4], M. Aschbacher [1], J. Saxl [6], and E. Bannai [2]. In this note we shall prove the following:

Theorem. Let $G$ be a 4-fold transitive permutation group on a set $\Omega$, where $|\Omega| = 5m + 4$ ($m$ integer) and greater than 5. If $H \neq 1$ is a normal subgroup of $G$, then $H$ is also 4-fold transitive on $\Omega$.

2. Proof of the theorem. In order to prove the theorem we need the following:

**Lemma.** Let $G$ be a $t$-fold transitive permutation group on a set $\Omega$ for $t \geq 4$ and let $H \neq 1$ be a normal subgroup of $G$. Then for all $A \subseteq \Omega$ with $|A| = t$, $H|_A = S_t$.

**Proof.** We omit the proof of the lemma. (See [3].)

Now we start with the proof of the theorem. Suppose $|\Omega| = 5m + 4$ be the minimal degree >5 such that there is a counterexample to the theorem. Let $G$ be a 4-fold transitive group on $\Omega$ of degree $5m + 4$ containing a non-trivial normal subgroup $H$ which is 3-fold transitive but not 4-fold transitive on $\Omega$. Let $\alpha, \beta, \gamma \in \Omega$ and let $\Gamma_1, \ldots, \Gamma_k$ be the $H_{\alpha, \beta, \gamma}$-orbits on $\Omega - \{\alpha, \beta, \gamma\}$. Then by assumption $k \geq 2$, $|\Gamma_1| = \cdots = |\Gamma_k|$ and $|\Gamma_i|$ is not divisible by 5. At first we shall show that there exists a non-trivial 5-element in $H$ fixing at least 4 points of $\Omega$. By the lemma above and a result of Wagner ([7], Lemma 4), $|\Gamma_i|$ must be even. Let $\delta \in \Gamma_i$, and let $T$ be a Sylow 2-subgroup of $H_{\alpha, \beta, \gamma}$. Then $T \neq 1$ by Theorem 2 of J. King [5]. Now if $T^g \subseteq G_{\alpha, \beta, \gamma}$ for some $g \in G$ then $T = G_{\alpha, \beta, \gamma} \cap H = H_{\alpha, \beta, \gamma}$ and hence there is an element $h$ of $H_{\alpha, \beta, \gamma}$ such that $T = T^h$. Thus by a well-known lemma of Witt $N_0(T)^{\langle x \rangle} T$ is 4-fold transitive.

1) I was informed that the same result was obtained also by E. Bannai (University of Tokyo) independently.
Since \(|\Gamma_1| = |H_{a|T}| \cdot |H_{a|T}| \) is even, \(N_H(T)^{fix T} \neq 1\). Then \(N_H(T)^{fix T}\) is 3-fold transitive by the theorem of Jordan, and using again the above mentioned theorem of King we see that \(|fix T| = 5, 7 \) or 11 and \(N_H(T)^{fix T}\) is \(S_n, A_7\), or \(M_{11}\). So \(5|N_H(T)^{fix T}|\). Hence there is an element \(y \in H\) which is of order 5. Since \(|\Omega| = 5m + 4, y\) fixes at least 4 points of \(\Omega\) and hence 5 divides \(|H_{a|T}|\). Let \(Q\) be a Sylow 5-subgroup of \(H_{a|T}\). Then by the same reason as for \(T\) the above \(N_O(Q)^{fix Q}\) is 4-fold transitive. Since \(|\Gamma_1|\) is not divisible by 5, \(Q\) is a Sylow 5-subgroup of \(H_{a|T}\) and hence by a lemma of Witt \(N_H(Q)^{fix Q}\) is 3-fold transitive, in particular \(N_H(Q)^{fix Q} \neq 1\). Clearly \(|fix Q| = 5a + 4 (a: \text{integer})\) and \(5a + 4 < 5m + 4\). Since \(N_H(Q)^{fix Q} \leq N_O(Q)^{fix Q}, N_H(Q)^{fix Q}\) is either 4-fold transitive on \(fix Q\) or 3-fold transitive on \(fix Q\) with \(|fix Q| = 4\). In either case we get the following: \(fix Q = \{\alpha, \beta, \gamma\} \subseteq \Gamma_1\). Thus \(Q\) has no fixed point on \(\Gamma_1\). Hence 5 divides \(|\Gamma_1| = |\Gamma_1|\), which is a contradiction. Thus we complete the proof of the theorem.

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References