189. On Spaces which Admit Closure-Preserving Covers by Compact Sets

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Let X be a T_1-space which admits a closure-preserving closed cover $\mathcal{F}$ by compact subsets. In [9], H. Tamano raised the question of whether or not such a space X must be paracompact. H. B. Potoczny gave in [6] a negative answer to this question, and proved in [7] that such a space X is paracompact whenever it is collectionwise normal.

In the same paper [7], he has stated that if each member of $\mathcal{F}$ is a finite subset then X is θ-refinable, and if, in addition, there is a positive integer n such that each member of $\mathcal{F}$ has no more than n points then X is metacompact. Moreover, he has conjectured that X must be always metacompact or θ-refinable without such severe restrictions.

In this paper, we shall give a solution to this problem.

Theorem 1. *If a T_1-space X has a closure-preserving closed cover $\mathcal{F}$ by compact subsets, then X is metacompact.*

It is known that a metacompact, collectionwise normal space is paracompact ([3] or [5]); consequently the above result of Potoczny follows immediately from our Theorem 1.

We need some lemmas to prove Theorem 1. A space X is said to be *almost expandable* [8], if for every locally finite collection $\{F_\alpha | \alpha \in A\}$ of subsets of X there exists a point-finite collection $\{G_\alpha | \alpha \in A\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for every $\alpha \in A$; every metacompact space is almost expandable ([8]). A cover $\mathcal{U}$ of a space is said to be *directed*, if for two members U and V of $\mathcal{U}$ there is a member W of $\mathcal{U}$ such that $U \cup V \subset W$.

The following lemma is an immediate consequence of [2, Theorem 2.2] (announced in [1]).

Lemma 1. *If every directed open cover of a space X has a cushioned refinement, then X is almost expandable.*

Lemma 2. *If a space X has a closure-preserving closed cover $\mathcal{F}$ by compact subsets, then X is almost expandable.*

Proof. Let $\mathcal{U}$ be a directed open cover of X. Since each member of $\mathcal{F}$ is compact, we can easily prove that $\mathcal{F}$ refines $\mathcal{U}$. Furthermore $\mathcal{F}$ is a closure-preserving closed cover, so that $\mathcal{F}$ is a cushioned refinement of $\mathcal{U}$ ([4]). Hence, by Lemma 1, X is almost expandable.

The following notations are due to [7]. Let $\mathcal{F}$ be a closure-preserving closed cover of a space X. For each point x of X, we put $K(x) = X - \cup\{F \in \mathcal{F} | x \notin F\}$. A point x is said to be *maximal* provided $K(x)$ is not properly contained in any $K(y)$. The set of all maximal points is denoted by $M(\mathcal{F})$. For a closed subset C of X, $\mathcal{F}_C = \{F \cap C | F \in \mathcal{F}\}$ is a closure-preserving closed cover of C. Then $M(\mathcal{F}_C)$ is the set of maximal points of C with respect to $\mathcal{F}_C$.

The following two lemmas were proved in [7].

Lemma 3. *If $\mathcal{F}$ is a closure-preserving closed cover of X by compact subsets, then $M(\mathcal{F})$ can be decomposed into a discrete collection of compact subsets.*

Lemma 4.*) *Let X be a T_1-space, and $\mathcal{F}$ a closure-preserving closed cover of X by compact subsets. Let $\{V_n | n=1, 2, \cdots\}$ be a sequence of open subsets of X such that $M(\mathcal{F}) \subset V_1$ and $M(\mathcal{F}_{C_n}) \subset V_{n+1}$, where $C_n = X - \cup\{V_i | i=1, \cdots, n\}$ for $n=1, 2, \cdots$. Then $\{V_n | n=1, 2, \cdots\}$ covers X.*

Lemma 5. *Let $\mathcal{F}$ be a closure-preserving closed cover of X by compact subsets, and let $\mathcal{U}$ be an open cover of X. Then for every closed subset C of X, there exists a point-finite collection $\mathcal{V}$ of open subsets of X which covers $M(\mathcal{F}_C)$ and which refines $\mathcal{U}$.*

Proof. By Lemma 3, in the subspace C, there is a discrete collection $\mathcal{K} = \{K_\alpha | \alpha \in A\}$ of compact subsets such that $\cup\{K_\alpha | \alpha \in A\} = M(\mathcal{F}_C)$. Since C is closed in X, $\mathcal{K}$ is also discrete in X. By Lemma 2, X is almost expandable. Hence there is a point-finite collection $\{G_\alpha | \alpha \in A\}$ of open subsets of X such that $K_\alpha \subset G_\alpha$ for each $\alpha \in A$. Now, since each K_α is compact, we have a finite subcollection $\mathcal{U}_\alpha$ of $\mathcal{U}$ covering K_α for each $\alpha \in A$. Then $\mathcal{V} = \{G_\alpha \cap U | U \in \mathcal{U}_\alpha, \alpha \in A\}$ is a point-finite collection of open subsets of X. It is obvious that $\mathcal{V}$ covers $M(\mathcal{F}_C)$ and refines $\mathcal{U}$.

Proof of Theorem 1. By Lemma 2, X is almost expandable, and hence, by [8], X is countably metacompact; i.e., every countable open cover of X has a point-finite open refinement. As is easily shown, a σ-point-finite open cover of a countably metacompact space has a point-finite open refinement. So, to complete the proof, it suffices to prove that every open cover of X has a σ-point-finite open refinement; this will be achieved by the same argument as in [7].

Now, let $\mathcal{U}$ be an open cover of X. By Lemma 5, we have a point-finite collection $\mathcal{V}_1$ of open subsets of X such that $\mathcal{V}_1$ refines $\mathcal{U}$ and $M(\mathcal{F}) \subset V_1$, where $V_1 = \cup\{V | V \in \mathcal{V}_1\}$. Let us put $C_1 = X - V_1$, then C_1 is a closed subset of X. Again, by Lemma 5, we have a point-finite col-

*) Potoczny [7, Lemma 5] stated that the lemma holds in the case where X is collectionwise normal, but this restriction is not necessary; indeed the collectionwise normality of X was not used in his proof.

lection $\mathcal{V}_2$ of open subsets of X such that $\mathcal{V}_2$ refines $\mathcal{U}$ and $M(\mathcal{F}_{C_1})\subset V_2$, where $V_2=\cup\{V|V\in\mathcal{V}_2\}$. If we put $C_2=X-(V_1\cup V_2)$, then C_2 is a closed subset of X. Continuing these processes we obtain a sequence $\{\mathcal{V}_n|n=1,2,\cdots\}$ of collections of open subsets of X such that each $\mathcal{V}_n$ is point-finite, each $\mathcal{V}_n$ refines $\mathcal{U}$, $M(\mathcal{F})\subset V_1$ and $M(\mathcal{F}_{C_n})\subset V_{n+1}$, where $V_n=\cup\{V|V\in\mathcal{V}_n\}$ and $C_n=X-(V_1\cup\cdots\cup V_n)$ for $n=1,2,\cdots$. By Lemma 4, the collection $\{V_n|n=1,2,\cdots\}$ covers X. Hence $\mathcal{V}=\cup\{\mathcal{V}_n|n=1,2,\cdots\}$ is a σ-point-finite open cover of X which refines $\mathcal{U}$. Thus the proof of Theorem 1 is completed.

Theorem 2. *If a T_1-space X has a σ-closure-preserving closed cover by compact subsets, then X is θ-refinable.*

Proof. By assumption, X has a closed cover $\mathcal{F}=\cup\{\mathcal{F}_n|n=1,2,\cdots\}$ each $\mathcal{F}_n$ of which is a closure-preserving collection of compact subsets. Let us put $F_n=\cup\{F|F\in\mathcal{F}_n\}$ for $n=1,2,\cdots$, then $\{F_n|n=1,2,\cdots\}$ is a closed cover of X and each F_n is metacompact by Theorem 1. The union of countably many closed metacompact (more generally, θ-refinable) subspaces is θ-refinable by [10, p. 824]. Hence X is θ-refinable.

It is known that a θ-refinable, collectionwise normal space is paracompact ([10]). Hence, as a corollary to Theorem 2, we have the following result proved in [7].

Corollary. *If a collectionwise normal T_1-space X has a σ-closure-preserving closed cover by compact subsets, then X is paracompact.*

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