181. *Cohomology of Vector Fields on a Complex Manifold*

By Toru TSUJISHITA
University of Tokyo

(Comm. by Kunihiko KODAIRA, M. J. A., Dec. 12, 1974)

§ 1. Let M be a complex manifold. Let $\mathcal{A}$ denote the space of smooth vector fields of type (1, 0) on M. $\mathcal{A}$ is regarded as a Lie algebra under the usual bracket operation. Recently it is shown that the Lie algebra structure of $\mathcal{A}$ uniquely determines the complex analytic structure of M (I. Amemiya [1]), and thus it would be interesting to calculate the cohomology of the Lie algebra $\mathcal{A}$ associated with various representations. In this note, we shall state some results concerning the cohomology of the Lie algebra $\mathcal{A}$. Details will appear elsewhere.

The author would like to express his gratitude to Professor K. Shiga for his valuable suggestions.

§ 2. We recall here briefly the definition of the cohomology group of a Lie algebra $\mathfrak{g}$ associated with a $\mathfrak{g}$-module W. Let $C^p(\mathfrak{g}; W)$ denote the space of alternating p-forms on $\mathfrak{g}$ with values in the vector space W for $p>0$; we put $C^0(\mathfrak{g}; W)=W$ and $C^p(\mathfrak{g}; W)=0$ for $p<0$. The coboundary operator $d: C^p(\mathfrak{g}; W)\to C^{p+1}(\mathfrak{g}; W)$ is defined by the following formula:

$$(d\omega)(X_1, \cdots, X_{p+1})=\sum_{i=1}^{p+1}(-1)^{i-1}X_i\omega(X_1, \cdots, \hat{X}_i, \cdots, X_{p+1})$$
$$+\sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{p+1})$$

$(X_1, \cdots, X_{p+1}\in \mathfrak{g}, \omega\in C^p(\mathfrak{g}; W))$. The p-th cohomology group of this cochain complex $C(\mathfrak{g}; W)=\oplus_p C^p(\mathfrak{g}; W)$ will be denoted by $H^p(\mathfrak{g}; W)$. If the $\mathfrak{g}$-module W has a ring structure such that $X(fg)=(Xf)g+f(Xg)$ $(X\in\mathfrak{g},\ f, g\in W)$, then the total cohomology $H^*(\mathfrak{g}; W)=\oplus_p H^p(\mathfrak{g}; W)$ has a graded ring structure. (For more details, see [3].)

§ 3. The Lie algebra $\mathcal{A}$ has a representation on the ring $\mathcal{F}$ of smooth functions on M when the vector fields are identified canonically with the derivations on the ring $\mathcal{F}$. We shall denote by $C^p_\Delta(\mathcal{A}; \mathcal{F})$ the subspace of $C^p_\partial(\mathcal{A}; \mathcal{F})$ consisting of the elements ω such that $\mathrm{supp}\,(\omega(X_1, \cdots, X_p))\subset\bigcap_{i=1}^p \mathrm{supp}\,(X_i)$ $(X_1, \cdots, X_p\in\mathcal{A})$. Furthermore we shall denote by $C^p_\partial(\mathcal{A}; \mathcal{F})$ the subspace of $C^p_\Delta(\mathcal{A}; \mathcal{F})$ consisting of the elements ω such that, if $f\in\mathcal{F}$ is anti-holomorphic on an open subset U of M, then $\omega(fX_1, X_2, \cdots, X_p)=f\omega(X_1, X_2, \cdots, X_p)$ on U for any $X_1, X_2, \cdots X_p\in\mathcal{A}$. If we put $C_\Delta(\mathcal{A}; \mathcal{F})=\oplus_p C^p_\Delta(\mathcal{A}; \mathcal{F})$, and $C_\partial(\mathcal{A}; \mathcal{F})=\oplus_p C^p_\partial(\mathcal{A}; \mathcal{F})$, then $C_\Delta(\mathcal{A}; \mathcal{F})$ and $C_\partial(\mathcal{A}; \mathcal{F})$ form a subcomplex of

$C(\mathcal{A};\mathcal{F})$ and $C_\Delta(\mathcal{A};\mathcal{F})$, respectively. The p-th cohomology group of the complex $C_\Delta(\mathcal{A};\mathcal{F})$ and $C_\partial(\mathcal{A};\mathcal{F})$ will, respectively, be denoted by $H^p_\Delta(\mathcal{A};\mathcal{F})$ and $H^p_\partial(\mathcal{A};\mathcal{F})$. We note that the total cohomology groups $H^*_\Delta(\mathcal{A};\mathcal{F})=\oplus_p H^p_\Delta(\mathcal{A};\mathcal{F})$ and $H^*_\partial(\mathcal{A};\mathcal{F})=\oplus_p H^p_\partial(\mathcal{A};\mathcal{F})$ are graded rings.

Theorem 1. *We have an isomorphism of graded rings:*

$$H^*_\partial(\mathcal{A};\mathcal{F})\cong H^*(M,\bar{\mathcal{O}})\otimes H^*(\mathfrak{gl}(n,\boldsymbol{C});\boldsymbol{C}),$$

where $\bar{\mathcal{O}}$ denotes the sheaf of germs of anti-holomorphic functions on M, and the complex dimension of M is denoted by n; $H^*(\mathfrak{gl}(n,\boldsymbol{C});\boldsymbol{C})$ is the cohomology of the Lie algebra $\mathfrak{gl}(n,\boldsymbol{C})$ associated with the trivial $\mathfrak{gl}(n,\boldsymbol{C})$-module $\boldsymbol{C}$.

If we denote by ι the inclusion homomorphism of cochain complexes: $C_\partial(\mathcal{A};\mathcal{F})\hookrightarrow C_\Delta(\mathcal{A};\mathcal{F})$, then we have

Theorem 2. *For $p\leq n$, ι induces an isomorphism:*

$$\iota^p: H^p_\partial(\mathcal{A};\mathcal{F})\longrightarrow H^p_\Delta(\mathcal{A};\mathcal{F}).$$

§ 4. Next, we shall consider the adjoint representation of the Lie algebra $\mathcal{A}$. As before we shall denote by $C^p_\Delta(\mathcal{A};\mathcal{A})$ the subspace of $C^p(\mathcal{A};\mathcal{A})$ consisting of the elements ω such that $\operatorname{supp}(\omega(X_1,\cdots,X_p))\subset\bigcap_{i=1}^p\operatorname{supp}(X_i)$ for all $X_1,\cdots,X_p\in\mathcal{A}$. Then $C_\Delta(\mathcal{A};\mathcal{A})=\oplus_p C^p_\Delta(\mathcal{A};\mathcal{A})$ is in fact a subcomplex of $C(\mathcal{A};\mathcal{A})$, and its cohomology will be denoted by $H^*_\Delta(\mathcal{A};\mathcal{A})=\oplus_p H^p_\Delta(\mathcal{A};\mathcal{A})$.

Theorem 3. *We have an isomorphism*

$$H^p_\Delta(\mathcal{A};\mathcal{A})\cong\bigoplus_{u+v+1=p}H^u(M,\bar\Theta)\otimes H^v(\mathfrak{gl}(n,\boldsymbol{C});\boldsymbol{C})\qquad\text{for } p\leq n,$$

where $\bar\Theta$ denotes the sheaf of germs of anti-holomorphic vector fields of type $(0,1)$ on M.

Corollary. *The quotient algebra of the Lie algebra of the derivations of $\mathcal{A}$ divided by the ideal of the inner derivations is isomorphic to the Lie algebra $H^0(M,\bar\Theta)$.*

Here $H^0(M,\bar\Theta)$ is regarded as a Lie algebra under the usual bracket operation of vector fields.

§ 5. We shall outline the proofs of the theorems.

From an obvious generalization of the Peetre's theorem (cf. [4]), we infer the following

Lemma 1. $C^p_\partial(\mathcal{A};\mathcal{F})\cong\Gamma(\bigwedge^p(JT)')$.

Here $JT=\varprojlim_k J^{k,0}(T)$ and $J^{k,0}(T)$ is the k-th holomorphic jet bundle of the holomorphic tangent bundle T, whose fiber over a point $x\in M$ is $\Gamma(T)/(I_x^{k+1}+\bar I_x)\Gamma(T)$, I_x and $\bar I_x$ being the ideals of the ring $\mathcal{F}$ generated, respectively, by such elements f as $(\partial f/\partial z^\alpha)(x)=0$ and $(\partial f/\partial\bar z^\alpha)(x)=0$ for $\alpha=1,\cdots,n$, with respect to a holomorphic coordinate system $\{z^1,\cdots,z^n\}$ around x.

Next, we shall introduce in $C_\partial(\mathcal{A};\mathcal{F})$ a filtration due to I. M.

Gel'fand and D. B. Fuks ([2]). Put $F_p C_{\bar\partial}^{p+q}(\mathcal{A}; \mathcal{F}) = \Gamma((\wedge^p T') \wedge (\wedge^q (JT)'))$, where T' and $(JT)'$ denote, respectively, the dual bundle of T and JT and $(\wedge^p T')\wedge(\wedge^q (JT)')$ denotes the subbundle of $\wedge^{p+q}(JT)'$ whose fiber over a point x of M is spanned by the elements of the form $\omega_1\wedge\cdots\wedge\omega_p\wedge\eta_1\wedge\cdots\wedge\eta_q$ $(\omega_i \in T'_x, \eta_j \in (JT)'_x)$. It follows then that $d(F_r C_{\bar\partial}^m(\mathcal{A}; \mathcal{F})) \subset F_r C_{\bar\partial}^{m+1}(\mathcal{A}; \mathcal{F})$, and we shall denote by $\{E_r^{p,q}, d_r^{p,q}\}$ the spectral sequence associated with this filtration, which is convergent to $H_{\bar\partial}^*(\mathcal{A}; \mathcal{F})$. Then we can show

Lemma 2. $E_2^{p,q} \cong H^p(M, \bar{\mathcal{O}}) \otimes H^q(\mathfrak{gl}(n, \boldsymbol{C}); \boldsymbol{C})$.

Using a formula in the Chern-Weil theory, we can prove the degeneracy of the spectral sequence:

Lemma 3. $d_r^{p,q} = 0$ *for* $r \geq 2$.

These two lemmas clearly imply Theorem 1.

In order to prove Theorem 2, we introduce a similar filtration in the complex $C_\Delta(\mathcal{A}; \mathcal{F})$ and compare the E_1-terms. Then the proof is reduced to the calculation of the cohomology of formal vector fields.

The proof of Theorem 3 is similar to that of Theorem 1.

§ 6. Finally we remark that Theorem 2 enables us to improve Theorem 3.9 of [4]. Let E be a complex smooth vector bundle over M and let $\Gamma(E)$ be the space of smooth cross-sections of E. $\Gamma(E)$ is called a *differential* $\mathcal{A}$*-module of connection type* if $\Gamma(E)$ is an $\mathcal{A}$-module such that $\operatorname{supp}(Xs) \subset \operatorname{supp}(X) \cap \operatorname{supp}(s)$ and $X(fs) = (Xf)s + f(Xs)$ holds for $X \in \mathcal{A}$, $s \in \Gamma(E)$, $f \in \mathcal{F}$. Then

Theorem 4. *If* $\Gamma(E)$ *is a differential* $\mathcal{A}$*-module of connection type, then* κ^* *(Chern* $(E)) = 0$, *where Chern* (E) *denotes the subring of* $H^*(M, \boldsymbol{C})$ *generated by the Chern classes* $c_i(E)$ $(i \geq 1)$ *and* $\kappa^* : H^*(M, \boldsymbol{C}) \to H^*(M, \bar{\mathcal{O}})$ *denotes the homomorphism induced by the inclusion map* $\kappa : \boldsymbol{C} \hookrightarrow \bar{\mathcal{O}}$.

References

[1] I. Amemiya: Lie algebra of vector fields and complex structure (to appear).

[2] I. M. Gel'fand and D. B. Fuks: Cohomologies of Lie algebras of vector fields with non-trivial coefficients. Functional Analysis and their Applications, **4**, 181–192 (1970).

[3] K. Shiga: Cohomology of Lie algebras over a manifold. I. Journ. Math. Soc. Japan, **26**, 324–361 (1974).

[4] K. Shiga and T. Tsujishita: Differential representations of vector field (to appear).