# 180. A Remark on q-conformally Flat Product Riemannian Manifolds

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Recently, the study of curvature structures of higher order has been developed by J. A. Thorpe, R. S. Kulkarni and many other people. Especially, Kulkarni has introduced the interesting double form *con* $\omega$ associated with the given double form $\omega$, which is a generalization of Weyl's conformal curvature tensor for the case of higher order. Also, the present first author has studied in [3] on $q$-conformal flatness for Riemannian manifolds.

The object of this paper is to investigate on the double forms in product Riemannian manifolds, and apply it to obtain a theorem on $q$-conformally flat product Riemannian manifolds. An exposition with detailed proof of Theorem 2 will be published elsewhere.

We shall assume, throughout this paper, that all manifolds are connected and all objects are of differentiability class $C^\infty$. For the terminology and notation, we generally follow [1] and [2].

**1.** In this section we shall give a brief summary of basic formulae for later use (for the details, see [2] or [3]).

Let $\Lambda^p(V)$ and $\Lambda^p(V^*)$ denote the exterior powers of a real $n$-dimensional vector space $V$ and its dual space $V^*$, respectively $(0\leqq p\leqq n)$. We consider the spaces

$$\mathcal{D}^{p,q}(V)=\Lambda^p(V^*)\otimes\Lambda^q(V^*),\quad 0\leqq p,q\leqq n,\quad \mathcal{D}(V)=\sum_{p,q=0}^{n}\mathcal{D}^{p,q}(V).$$

An element $\omega\in\mathcal{D}^{p,q}(V)$ is called *double form of type* $(p,q)$ *on* $V$, and its value on $u=x_1\wedge x_2\wedge\cdots\wedge x_p\in\Lambda^p(V)$ and $v=y_1\wedge y_2\wedge\cdots\wedge y_q\in\Lambda^q(V)$ is denoted by

$$\omega(u\otimes v)=\omega(x_1x_2\cdots x_p\otimes y_1y_2\cdots y_q).$$

$\mathcal{D}(V)$ forms an associative ring with respect to the natural "exterior multiplication $\wedge$", and we have

$$\omega\wedge\theta=(-1)^{pr+qs}\theta\wedge\omega \tag{1}$$

for any double forms $\omega,\theta$ of types $(p,q)$, $(r,s)$, respectively. A symmetric double form of type $(p,p)$ is called the *curvature structure of order* $p$ *on* $V$, and the set of such elements is denoted by $C^p(V)$. $C(V)=\sum_{p=0}^{n}C^p(V)$ forms a commutative subring of $\mathcal{D}(V)$ called the *ring of*

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*curvature structures* on $V$.

Let $g \in \mathcal{C}^1(V)$ be a metric on $V$. The *contraction* $c$ maps $\mathcal{D}^{p,q}(V)$ into $\mathcal{D}^{p-1,q-1}(V)$. If $\omega \in \mathcal{D}^{p,q}(V)$ and $p=0$ or $q=0$, we set $c\omega=0$. If both $p, q \geqq 1$, then we set

$$(2) \qquad c\omega(x_1 \cdots x_{p-1} \otimes y_1 \cdots y_{q-1}) = \sum_{k=1}^{n} \omega(e_k x_1 \cdots x_{p-1} \otimes e_k y_1 \cdots y_{q-1}),$$

where $\{e_1, e_2, \cdots, e_n\}$ is an orthonormal base for $V$. Then, for any double form $\omega$ of type $(p, q)$ we have

$$c(g^{r+1} \wedge \omega) = g^{r+1} \wedge c\omega + (r+1)(n-p-q-r) g^r \wedge \omega \qquad (r \geqq 0),$$

from which we obtain inductively

$$(3) \qquad c^s g^t = \frac{t\,!\,(n-t+s)\,!}{(t-s)\,!\,(n-t)\,!} g^{t-s}$$

for any integers $s, t$ satisfying $0 \leqq s \leqq t \leqq n$.

**2.** Let $V_a$ be a real $n_a$-dimensional vector space $(a=1,2)$. Let us set $V=V_1 \oplus V_2$ and identify $V_a$ with the subspace of $V$. An element $\alpha \in \Lambda^p(V^*)$ is called *of type* $(p_1, p_2)$ if $p=p_1+p_2$ and, for vectors $x_i$ in $V_1$ or in $V_2$, $\alpha(x_1, \cdots, x_p)=0$ except for the case when the $p_1$ vectors $x_i$ belong to $V_1$ and the other $p_2$ vectors $x_j$ belong to $V_2$. The set of such elements is indicated by $\Lambda^{p_1,p_2}(V^*)$. Then we have

$$\Lambda^p(V^*) = \sum_{p_1+p_2=p} \Lambda^{p_1,p_2}(V^*) \qquad \text{(direct sum).}$$

Now, we consider the spaces

$$\mathcal{D}^{(p_1,q_1;p_2,q_2)}(V) = \Lambda^{p_1,p_2}(V^*) \otimes \Lambda^{q_1,q_2}(V^*),$$

and we call an element $\omega$ of $\mathcal{D}^{(p_1,q_1;p_2,q_2)}(V)$ the double form of type $(p_1, q_1; p_2, q_2)$ on $V$. Then we have

$$\mathcal{D}^{p,q}(V) = \sum_{p_1+p_2=p} \sum_{q_1+q_2=q} \mathcal{D}^{(p_1,q_1;p_2,q_2)}(V) \qquad \text{(direct sum).}$$

Also, we can identify

$$\mathcal{D}^{p_1,q_1}(V_1) = \mathcal{D}^{(p_1,q_1;0,0)}(V), \qquad \mathcal{D}^{p_2,q_2}(V_2) = \mathcal{D}^{(0,0;p_2,q_2)}(V).$$

Let $g_1, g_2$ be metrics on the vector spaces $V_1, V_2$, respectively. We introduce a metric $g$ on $V$ by the formula

$$g(u \otimes v) = g_1(u_1 \otimes v_1) + g_2(u_2 \otimes v_2),$$

where $u_a, v_a$ are $V_a$-components of $u, v \in V$, respectively. Also, we define two mappings $c_1, c_2 : \mathcal{D}^{p,q}(V) \to \mathcal{D}^{p-1,q-1}(V)$ as follow : If $\omega \in \mathcal{D}^{p,q}(V)$ and $p=0$ or $q=0$, we set $c_1\omega = c_2\omega = 0$. If both $p, q \geqq 1$, we set

$$(4) \qquad \begin{aligned} c_1\omega(x_1 \cdots x_{p-1} \otimes y_1 \cdots y_{q-1}) &= \sum_{i=1}^{n_1} \omega(f_i x_1 \cdots x_{p-1} \otimes f_i y_1 \cdots y_{q-1}), \\ c_2\omega(x_1 \cdots x_{p-1} \otimes y_1 \cdots y_{q-1}) &= \sum_{j=1}^{n_2} \omega(h_j x_1 \cdots x_{p-1} \otimes h_j y_1 \cdots y_{q-1}) \end{aligned}$$

where $\{f_1, \cdots, f_{n_1}\}$ and $\{h_1, \cdots, h_{n_2}\}$ are orthonormal bases for $V_1$ and $V_2$, respectively. It is easy to see that

$$(5) \qquad c = c_1 + c_2 \qquad \text{on } \mathcal{D}(V),$$

and for any double forms $\omega, \theta$ of types $(p_1, q_1; 0, 0)$, $(0, 0; p_2, q_2)$, respectively, we have

(6) $$c_2\omega=0 \quad\text{and}\quad c_1\theta=0.$$

**Theorem 1.** *For any double forms $\omega, \theta$ of types $(p_1, q_1; 0, 0)$, $(0, 0; p_2, q_2)$, respectively, we have*

$$c(\omega\wedge\theta)=c_1\omega\wedge\theta+(-1)^{p_1+q_1}\omega\wedge c_2\theta.$$

**Proof.** Let $Sh(r, s)$ denote the set of all $(r, s)$-shuffles

$$Sh(r, s)=\{\tau\in S_{r+s};\ \tau_1<\cdots<\tau_r \text{ and } \tau_{r+1}<\cdots<\tau_{r+s}\},$$

$S_{r+s}$ being the symmetric group of degree $r+s$. Then, from the assumptions of Theorem 1 and (4), we find

$$\begin{aligned}c_1(\omega\wedge\theta)&(x_2\cdots x_{p_1+p_2}\otimes y_2\cdots y_{q_1+q_2})\\ &=\sum_{i=1}^{n_1}\sum_{\alpha,\beta}\varepsilon_\alpha\varepsilon_\beta\omega(f_ix_{\alpha(2)}\cdots x_{\alpha(p_1)}\otimes f_iy_{\beta(2)}\cdots y_{\beta(q_1)})\\ &\quad\times\theta(x_{\alpha(p_1+1)}\cdots x_{\alpha(p_1+p_2)}\otimes y_{\beta(q_1+1)}\cdots y_{\beta(q_1+q_2)})\\ &=(c_1\omega\wedge\theta)(x_2\cdots x_{p_1+p_2}\otimes y_2\cdots y_{q_1+q_2}),\end{aligned}$$

where the second summation is taken over all shuffle-permutations $\alpha\in Sh(p_1-1, p_2)$ and $\beta\in Sh(q_1-1, q_2)$, and $\varepsilon_\alpha, \varepsilon_\beta$ denote the sign of the respective permutations $\alpha, \beta$. Similarly, we see that $c_2(\theta\wedge\omega)=c_2\theta\wedge\omega$. Thus, Theorem 1 follows from the equations (1) and (5). q.e.d.

**Corollary 1.** *For any curvature structures $\omega\in \mathcal{C}^p(V_1)$ and $\theta\in\mathcal{C}^q(V_2)$, we have*

(7) $$c^r(\omega\wedge\theta)=\sum_{k=0}^{r}{}_rC_k c_1^{r-k}\omega\wedge c_2^k\theta.$$

**3.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold and $T_m(M)$ be its tangent space at a point $m\in M$. The vector bundles $\mathcal{D}^{p,q}(M)$ and $\mathcal{C}^p(M)$ assign the vector spaces $\mathcal{D}^{p,q}(T_m(M))$ and $\mathcal{C}^p(T_m(M))$, respectively, as fibres to each point $m\in M$. The algebraic notions and operations in section 1 can be applied to the rings

$$\tilde{\mathcal{D}}(M)=\sum_{p,q=0}^{n}\tilde{\mathcal{D}}^{p,q}(M), \qquad \tilde{\mathcal{C}}(M)=\sum_{p=0}^{n}\tilde{\mathcal{C}}^p(M),$$

where $\tilde{E}$ denotes the vector space of all global sections of the bundle $E$. Let $R\in\tilde{\mathcal{C}}^2(M)$ be the curvature tensor field of type $(0, 4)$ on $M$. The manifold $(M, g)$ is called *$q$-conformally flat* if $n>4q-1$ and $con\,R^q=0$, where

(8) $$con\,R^q=R^q+\sum_{k=1}^{2q}\frac{(-1)^kg^k\wedge c^kR^q}{k!\prod_{j=0}^{k-1}(n-4q+2+j)}.$$

Now, let $(M, g)$ be a product Riemannian manifold of two Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$ with dimensions $n_1$ and $n_2$, respectively. Then the tangent space $T_m(M)$ at each point $m=(m_1, m_2)$ $(m_1\in M_1, m_2\in M_2)$ is isomorphic in a natural way to the direct sum $T_{m_1}(M_1)\oplus T_{m_2}(M_2)$, so we identify

$$T_m(M)=T_{m_1}(M_1)\oplus T_{m_2}(M_2).$$

Also, the metric $g$ and the curvature tensor $R$ are given by

$$g(m)(u\otimes v)=g_1(m_1)(u_1\otimes v_1)+g_2(m_2)(u_2\otimes v_2),$$

(9) $$R(m)(uv\otimes xy)=R_1(m_1)(u_1v_1\otimes x_1y_1)+R_2(m_2)(u_2v_2\otimes x_2y_2),$$

at each point $m=(m_1, m_2)$, respectively, where $R_a \in \tilde{C}(M_a)$ is the curvature tensor of $M_a$ and $u_a, v_a, x_a, y_a$ are the $T_{m_a}(M_a)$-components of $u, v, x, y \in T_m(M)$, respectively. Thus, all algebraic operations and rules mentioned in the previous section can be now re-formulated for the manifolds $M$ and $M_a$ $(a=1,2)$.

**Theorem 2.** *Let $(M,g)$ be a product Riemannian manifold of two Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$ with constant sectional curvatures $\kappa_1$ and $\kappa_2$, respectively. Suppose that both $M_1$ and $M_2$ are of dimension $\geqq 2q$ $(q \geqq 1)$. Then, a necessary and sufficient condition for $(M,g)$ to be q-conformally flat is*

$$\kappa_1+\kappa_2=0. \tag{10}$$

*Outline of the proof.* We set $\dim M_a = n_a$ $(a=1,2)$. By the assumptions of Theorem 2 and the formula (9), we have

$$R_a=\frac{1}{2}\kappa_a g_a^2 \quad (a=1,2), \qquad R=\frac{1}{2}(\kappa_1 g_1^2+\kappa_2 g_2^2). \tag{11}$$

Substitute this into the formula (8), and then apply the equations (3), (5), (6) and (7) to the resulting equation. Then, after long but straightforward calculations, we find that the component of type $(2q, 2q; 0, 0)$ of $con\, R^q$ is given by the formula

$$2^{-q}\prod_{j=0}^{2q-1}\frac{n_2-j}{n-4q+2+j}(\kappa_1+\kappa_2)^q g_1^{2q}.$$

Thus we get (10). Conversely, it is well-known that (10) and (11) imply that $con\, R=0$, that is, $(M,g)$ is conformally flat. Hence, we have $con\, R^q=0$ (cf. Theorem 1 in [3]).

**Remark.** The assumption that both $M_1$ and $M_2$ are of dimension $\geqq 2q$ is essential in Theorem 2. In fact, suppose that $(M_1, g_1)$ is an arbitrary Riemannian manifold of dimension $n_1<2q$ and $(M_2, g_2)$ is a flat Riemannian manifold of dimension $n_2>4q-n_1-1$, then $R^q=0$ by (9), hence $(M,g)$ is always $q$-conformally flat.

**Corollary 2.** *Under the assumptions in Theorem 2, $(M,g)$ is q-conformally flat if and only if $(M,g)$ is conformally flat in usual sense.*

## References

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