7. *On a Property of Quadratic Farey Sequences*

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(Comm. by Kenjiro SHODA, M. J. A., Jan. 13, 1975)

§ 1. Introduction and notations. The well known Farey sequence of order s on [0, 1] is in reality an ordered list of all zeros of linear polynomials $ax-b$ with integral coefficients satisfying $0 \leqq b \leqq a \leqq s$. The quadratic Farey sequence of order s is defined as ordered list of all the real roots of the equation $ax^2+bx+c=0$, which $0 \leqq a \leqq s$, $|b| \leqq s$, $0 \neq |c| \leqq s$. Recently, H. Brown and K. Mahler study the quadratic Farey sequence on [0, 1], and give some data via the computer [1]. In this paper, we give a formula to Table II, [1], i.e. the value of the determinant formed by the coefficients of three consecutive quadratics at certain rational points.

In this paper itallic letters and letters with a suffix or sign, r_p^*, l_p etc. denote all integers except x, y. The symbol $[q/p]$ denotes the integral part of q/p; that is, the integer such that $[q/p] \leqq q/p < [q/p]+1$. Put

$L_s = \{(a, k, l) : s \geqq a \geqq 0,\ 0 \neq |l| \leqq s,\ |k| \leqq s\}$

$N_{s,r}^+ = \{(l, k) : nl-mk=r,\ 0<l \leqq s,\ |k| \leqq s\}$

$N_{s,r}^- = \{(l, k) : nl-mk=r,\ 0>l \geqq -s,\ |k| \leqq s\}$

$d(a, r, k, l) = d_{m/n}(a, r, k, l) = |(m/n, m/n)| - |$the point which (1) $y=l/(ax+k)$ intersects with (2) $y=x|$, where $|*|$ denotes the length of a vector $*$. Now we denote an order to the set $M_{s,r}$, where $M_{s,r}=N_{s,r}^+$ or $N_{s,r}^-$. If $M_{s,r} \neq \emptyset$, $(l, k)<(l', k') \Leftrightarrow |l|<|l'|$ and $(l, k)=(l', k') \Leftrightarrow l=l'$. Here we call (l, k) or l maximum in $M_{s,r}$ when the value $|l|$ is maximum among the element $(l, k) \in M_{s,r}$.

In order to obtain the results, we consider fractional functions (1) $y=l/(ax+k)$ for $(a, k, l) \in L_s$ and the equation (2) $y=x$. Then, the set M_s of all the positive points on [0, 1] which (1) is intersecting with (2) gives the quadratic Farey sequence of order s. The necessary and sufficient condition that (1) throws the point $(m/n, m/n)$ is $a=nr/m^2$, where $r=nl-mk$, but $a=nr/m^2$ is not necessary integral number, so, we must find the fractional function (1) with integral coefficients throwing the nearest point to $(m/n, m/n)$. That is, it is reduced to find two elements $(a, k, l) \in L_s$ such that $d(a, r, k, l)>0$ is minimum and $d(a, r, k, l)<0$ is maximum. Here we call the equation giving this nearest point smaller (larger) than m/n lower (upper) best approximating equation with respect to m/n. Our results are given as Theorems 1–3.

§ 2. Results. Theorem 1. $m \geqq 2$, $(n, m)=1$ $(t=-1, 1)$
Let r_t be maximum integers satisfying $nr_t \equiv t(m^2)$, $(nr_t - t)/m^2 \leqq s$, and let (l_t, k_t) be maximum in N^+_{s, r_t}. Then, 1) *the upper best approximating primitive equation is* (3) $((nr_1 - 1)/m^2)x^2 + k_1x - l_1 = 0$, *and the lower best approximating primitive equation is* (4) $((nr_{-1}+1)/m^2)x^2 + k_{-1}x - l_{-1} = 0$. 2) *the value L of the determinant of the matrix of the coefficient of the equation* (3), $nx - m = 0$, *and* (4) *is* $L = -(r_1 + r_{-1})/m^2$.

Corollary. $(t=-1, 1)$

1) *if* $([s/n]-1)m^2 < r_t < [s/n]m^2$, *then* $L = 1 - 2[s/n]$

2) *if* $([s/n]-1)m^2 < \min(r_1, r_{-1}) < [s/n]m^2 < \max(r_1, r_{-1}) < ([s/n]+1)m^2$, *then* $L = -2[s/n]$

3) *if* $[s/n]m^2 < r_t$, *then* $L = -1 - 2[s/n]$

4) *the value L is always negative.*

Theorem 2.

1) *if* $m=1$, *then* $L = -\{[(s+1)/n] + [(s-1)/n]\}$

2) *if* $m=n$, *then* $L = -2s$

Theorem 3. $n \geqq 3$
The absolute values of L are symmetric with respect to rational member 1/2.

Theorem 1 is easily obtained from the following main lemma and the lemma.

Main lemma. *Let the condition of Theorem* 1 *hold. Then,* 1) $N^+_{s, r_t} \neq \emptyset$, $((nr_t - t)/m^2) \neq 0$. 2) $\min_{L_s, d>0} d(a, r, k, l) = d((nr_{-1}+1)/m^2, r_{-1}, k_{-1}, l_{-1})$, $\max_{L_s, d<0} d(a, r, k, l) = d((nr_{+1}-1)/m^2, r_1, k_1, l_1)$.

Lemma. $(t=-1, 1)$

1) *Equations* $((nr_t - t)/m^2)x^2 + k_tx - l_t = 0$ *are primitive.*

2) *The above equations don't have rational roots.*

In the next § 3, we give only the sketch of the proof of the main lemma. Because the method of the proof is all elementary, and it needs many pages for the proof. And the proof of the lemma and Theorem 3 is easily obtained, and the proof of the corollary is also, noting $([s/n]-1)m^2 < r_t < ([s/m^2]+1)m^2$, and $r_1 + r_{-1} \equiv 0(m^2)$.

§ 3. The sketch of the proof of the main lemma. The results of the part I of the main lemma are clear. The proof of the part II takes the following steps.

Step I. The element r of the set $\{r : r = nl - mk$, where $0 \neq |l| \leqq s$, $|k| \leqq s\}$ takes positive value, negative value or zero. But we believe that we may restrict ourselves to $r \geqq 0$. Since, if $r<0$, $nr \equiv p(m^2)$, then there exists $(a^*, r^*, k^*, l^*) \in L_s$ such that $0 < d(a^*, r^*, k^*, l^*) \leqq d(a, r, k, l)$ if $l>0$, or $0 > d(a^*, r^*, k^*, l^*) \geqq d(a, r, k, l)$ if $l<0$. Where $k^* = -k$, $l^* = -l$, $a^* = \min((n(-r)-(m^2-p))/m^2, s)$, $r^* = -r > 0$.

Step II. When $0 \leqq nr_p/m^2 \leqq s+1$, $r_p = nl_p - mk_p$, $r_t > 0$, $(t=-1, 1)$ $nr_p \equiv p(m^2)$, $N^+_{s, r_p} \neq \emptyset$, (l^*_p, k^*_p) maximum in N^+_{s, r_p}.

Then, 1) $r_p \leqq pr_1$, $l_p^* \leqq pl_1^*$ $(p=1, 2, \cdots, m^2)$

2) $r_p \leqq (m^2-p)r_{-1}$, $l_p^* \leqq (m^2-p)l_{-1}^*$ $(p=0, 1, \cdots, m^2-1)$ similarly, when $|r_p - r_t| < m^2$ if $r_p \geqq m^2$, $0 < r_t < m^2$ if $0 \leqq r_p < m^2$, $N_{s,r_p}^- \neq \emptyset$, (l_p^*, k_p^*) maximum in N_{s,r_p}^-.

Then, 1′) $r_p \leqq (m^2-p)r_{-1}$, $(-l_p^*) \leqq (m^2-p)l_1^*$ $(p=0, 1, \cdots, m^2-1)$

2′) $r_p \leqq pr_1$, $(-l_p^*) \leqq pl_{-1}^*$ $(p=1, 2, \cdots, m^2)$.

Here, we consider the value of the fractional equation (1) $y = l/(ax+k)$ for $x = m/n$ and the value of x for $y = m/n$. Then, we obtain from (1),

$$y(a, k, l) = y_{m/n}(a, k, l) = nl/(nk+am),$$

$$x(a, k, l) = x_{m/n}(a, k, l) = \begin{cases} r/am & \text{for } a \neq 0 \\ \pm\infty & \text{for } a = 0. \end{cases}$$

Roughly speaking, to find min $d(a, r, k, l) > 0$ (Max $d(a, r, k, l) < 0$) is reduced to find the maximum (minimum) value $y(a, k, l)$ smaller (larger) than m/n and the maximum (minimum) value $x(a, k, l)$ smaller (larger) than m/n. So, in this estimation, we use the above inequalities 1, 2, 1′, 2′. For example, $y((nr_p - p)/m^2, k_p^*, l_p^*) - y((nr_p - p)/m^2, k_1^*, l_1^*) = nml_p^*/(n^2 l_p^* - p) - nml_1^*/(n^2 l_1^* - 1) = nm(pl_1^* - l_p^*)/(n^2 l_p^* - p)(nl_1^* - 1) \geqq 0$ by enequality (1). Thus we find min $d(a, r, k, l) > 0$ (max $d(a, r, k, l) < 0$) is obtained by $a = (nr_{-1}+1)/m^2$ $(a = (nr_1 - 1)/m^2)$, $k = k_{-1}^* (= k_1^*)$, $l = l_{-1}^* (= l_1^*)$.

Step III. Let $nr_t \equiv t(m^2)$, (l_t, k_t) maximum in N_{s,r_t}^+ and $r_t^* > 0$ maximum integer such that $(nr_t^* - t)/m^2 \leqq s$, $nr_t^* \equiv t(m^2)$, (l_t^*, k_t^*) maximum in $N_{s,r_t^*}^+$, $0 < r_t < r_t^*$ $(t = -1, 1)$. Then $y((nr_1 - 1)/m^2, k_1, l_1) \geqq y((nr_1^* - 1)/m^2, k_1^*, l_1^*) \geqq m/n$, $x((nr_1 - 1)/m^2, k_1, l_1) \geqq x((nr_1^* - 1)/m^2, k_1^*, l_1^*) > m/n$ and $m/n > y((nr_{-1}^* + 1)/m^2, k_{-1}^*, l_{-1}^*) \geqq y((nr_{-1} + 1)/m^2, k_{-1}, l_{-1})$, $m/n > x((nr_{-1}^* + 1)/m^2, k_{-1}^*, l_{-1}^*) \geqq x((nr_{-1} + 1)/m^2, k_{-1}, l_{-1})$. By Step II, if $0 \leqq nr/m^2 < s$, min $d(a, r, k, l) > 0$ (max $d(a, r, k, l) < 0$) is obtained when $r = r_{-1}^*$ $(r = r_1^*)$.

Step IV. We find easily that if $nr/m^2 \geqq s$, $l > 0$, we obtained that $d(s, r, k, l) \geqq d((nr_1^* - 1)/m^2, r_1^*, k_1^*, l_1^*) > 0$, where $r_1^* > 0$ maximum integer such that $0 < (nr_1^* - 1)/m^2 \leqq s$, $nr_1^* \equiv 1(m^2)$, and if $nr/m^2 \geqq s$, $l < 0$, then $|d(s, r, k, l)| \geqq |d((nr_t - t)/m^2, r_t, k_t^*, l_t^*)|$.

Thus, the main lemma holds.

Example. To find the upper (lower) best approximating equation with respect to 2/5.

1. In the case of order $s = 5$

$r_{-1} = 3$ is maximum integer satisfying $(5r+1)/4 \leqq 5$, $5r \equiv -1$ (4)

$r_{+1} = 1$ is maximum integer satisfying $(5r-1)/4 \leqq 5$, $5r \equiv 1$ (4)

so $L = -((r_1 + r_{-1})/4) = -1$ by Theorem 1,

and

$(l_{-1}^*, k_{-1}^*) = (1, 1)$ is maximum in $N_{5,r_{-1}}^+ = \{(l, k) : 3 = 5l - 2k \ \ 0 < l \leqq 5, |k| \leqq 5\}$

$(l_{+1}^*, k_{+1}^*) = (1, 2)$ is maximum in $N_{5,r_{+1}}^+ = \{(l, k) : 1 = 5l - 2k, 0 < l \leqq 5, |k| \leqq 5\}$

$(nr_{-1}+1)/m^2=4$, $(nr_{+1}-1)/m^2=1$,

so

$$4x^2+x-1=0 \text{ (lower)}$$
$$x^2+2x-1=0 \text{ (upper)}$$

2. In the case of order $s=6$

$r_{-1}=3$ is maximum integer satisfying $(5r_{-1}+1)/4\leqq 6$, $5r_{-1}\equiv -1$ (4)

$r_{+1}=5$ is maximum integer satisfying $(5r_{+1}-1)/4\leqq 6$, $5r_{+1}\equiv 1$ (4)

$0<r_{-1}<4<r_{+1}<8$

by corollary, so $L=-2$

and

$$(l^*_{-1}, k^*_{-1})=(3, 6) \text{ is maximum in } N^+_{6,r_{-1}}$$
$$(l_{+1}, k^*_{+1})=(3, 5) \text{ is maximum in } N^+_{6,r_{+1}}$$

so

$$4x^2+6x-3=0 \text{ (lower)}$$
$$6x^2+5x-3=0 \text{ (upper).}$$

Acknowledgement. The author wishes to thank Professor T. Tsuzuku for his kind suggestions during the preparation of this paper.

Reference

[1] Brown, H., and Mahler, K.: A generalization of Farey sequence. Some exploration via the cumputer. J. Number Theory, **3**, 364–370 (1971).