6. *A Note on Partially Hypoelliptic Operators*

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1. Introduction. We shall study in this note the hypoellipticity of the following partial differential operator,

(1.1) $P(t; D_x, \partial_t) = \partial_t + ait^{l_0}D_x^m + bt^{l_1}D_x^{2n}, \qquad (a, b \in \boldsymbol{R}, i=\sqrt{-1}),$

where $\partial_t = \partial/\partial t$, $D_x = -i\partial/\partial x$ and $(x, t) \in \boldsymbol{R}_x \times (-1, 1)$.

Concerning hypoelliptic operators various studies have been made by many authors. One of the recent developments is that of degenerate operators. In this case almost studies are concentrated in the relation between the order of derivative and that of degeneracy of the coefficient, and there arise interesting properties which do not occur in the regular case. The difficulties lie on how to be dissolved the singularity appeared on a submanifold (or a subset) where the operator degenerates (see [1]~[9] and those references).

Contrary to this point of view, our purpose in this note is to show that under some conditions the operator (1.1) is regular (in some sense) on $t=0$, but is not regular on $t=t_0\neq 0$.

Let us now present an exact statement of our result. For this purpose we assume,

(1.2)
$$\begin{cases} \text{(i)} & m>2n, \\ \text{(ii)} & l_0 \text{ and } l_1 \text{ are a non-negative integer and a non-negative} \\ & \text{even integer respectively,} \\ \text{(iii)} & a\cdot b\neq 0, \\ \text{(iv)} & (m-1)/(l_0+1)<2n/(l_1+1). \end{cases}$$

Then we have

Theorem. *Under the assumptions* (1.2) *the operator given by* (1.1) *has the following properties;*

(i) *P and its adjoint ${}^tP$ are hypoelliptic on $t=0$ with respect to x, i.e., if $Pu \in C^\infty(I_x \times J_t)$ and $u \in \mathcal{E}^0(J_t; \mathcal{D}'(I_x))$, then $u(x, 0) \in C^\infty(I_x)$, where $I_x = (-\alpha, \alpha)$, $J_t = (-\beta, \beta)$. It also holds for ${}^tP$.*

(ii) *P and ${}^tP$ are not hypoelliptic on $t=t_0\neq 0$ with respect to x.*

Remark. (i) If m, l_0 and l_1 are even integers, $\mathrm{Re}\, ai>0$ and $\mathrm{Re}\, b>0$ (or if m and l_0 are even integers, $\mathrm{Re}\, ai>0$ and $m/(l_0+1) \geqq 2n/(l_1+1)$), then P and ${}^tP$ are hypoelliptic in $\boldsymbol{R}_x \times (-1, 1)$.

(ii) If m is an even integer, l_0 and l_1 are odd integers, $\mathrm{Re}\, ai>0$ and $\mathrm{Re}\, b>0$ (or if m is an even integer, l_0 is an odd integer, $\mathrm{Re}\, ai>0$

and $m/(l_0+1)\geqq 2n/(l_1+1)$), then ${}^tP$ is hypoelliptic in $\boldsymbol{R}_x\times(-1,1)$ but P is not hypoelliptic on $t=0$.

(iii) We can easily show that if in (1.1) l_1 is an odd integer and $b>0$, then P is not hypoelliptic on $t=0$. Moreover in this case if we assume (1.2)–(i), (iv) then ${}^tPu\in C^\infty(I_x\times J_t)$ implies $u(x,0)\in C^\infty(I_x)$ when $u(x,t)\in\mathcal{E}^0(J_t;\mathcal{D}'(I_x))$. This is proved by combining our method and that of Y. Kannai [4] or Y. Kato [6].

(iv) In the assumptions (1.2) the condition (iv) is necessary to obtain our result (see Remark in Section 3).

2. Proof of Theorem-(i). Without loss of generality we may assume that $b>0$. And also it suffices to show the theorem for the operator ${}^tP$. We note that our theorem is proved by constructing a very regular fundamental solution on $t=0$ (in some sense). So let us recall that if $F(x,t,s)$ satisfies

$$P(t;D_x,\partial_t)F(x,t,s)=\delta(x,t-s), \tag{2.1}$$

then $F(x-y,t,s)$ is a fundamental solution of P. In our case $F(x,t,s)$ is given by

$$F(x,t,s)=H(t-s)\mathcal{F}_\xi^{-1}[\hat{E}(\xi,t,s)], \tag{2.2}$$

where $H(t)$ is Heaviside's function and $\hat{E}$ is given by

$$\hat{E}(\xi,t,s)=\exp\left[-\int_s^t \mathcal{L}(\tau,\xi)d\tau\right],\qquad \mathcal{L}(t,\xi)=ait^{l_0}\xi^m+bt^{l_1}\xi^{2n}. \tag{2.3}$$

At first, let us evaluate $\partial_{t,s}^l\partial_\xi^\mu\hat{E}$. It is obvious that $\partial_{t,s}^l\partial_\xi^\mu\hat{E}$ is expressed by a linear combination of terms

$$\hat{E}(\xi,t,s)\cdot\prod_{i\in I}\partial_\xi^{\mu_i}\int_s^t\mathcal{L}(\tau,\xi)d\tau\cdot\prod_{j\in J}\partial_t^{l_j}\partial_\xi^{\mu_j}\mathcal{L}(t,\xi)\cdot\prod_{k\in K}\partial_s^{l_k}\partial_\xi^{\mu_k}\mathcal{L}(s,\xi), \tag{2.4}$$

where $\sum_i\mu_i+\sum_j\mu_j+\sum_k\mu=\mu_k$, $\#I+\#J+\#K\leqq\mu+l$, $\#J+\#K\leqq l$, and $\mu_i\geqq 1$ ($\#I$ denotes the number of elements in I). Therefore it is obvious that

$$|\partial_{t,s}^l\partial_\xi^\mu\hat{E}|\leqq C(1+|\xi|)^{m(l+\mu)-\mu}\exp[-\varepsilon(t^{l_1+1}-s^{l_1+1})|\xi|^{2n}], \tag{2.5}$$

for some positive constants C and ε. But in the case where $s=0$ or $t=0$ we can obtain

Lemma 1. *Let $\delta_1=\min\{1,(l_0+1)(2n/(l_1+1)-(m-1)/(l_0+1))\}$, then we have*

$$|\partial_t^l\partial_\xi^\mu\hat{E}(\xi,t,0)|\leqq C(1+|\xi|)^{ml-\mu\delta_1}\exp[-\varepsilon t^{l_1+1}|\xi|^{2n}],\qquad t\geqq 0. \tag{2.6}$$

Moreover if $\delta_1\mu>ml$ it holds that

$$|\partial_t^l\partial_\xi^\mu\hat{E}|\rightrightarrows 0\qquad as\ t\searrow 0, \tag{2.7}$$

where $\rightrightarrows$ means uniformly convergence in ξ. It also holds for $\partial_s^l\partial_\xi^\mu\hat{E}(\xi,0,s)$, $s\leqq 0$.

Proof. It suffices to show our assertions for the term (2.4) (putting $s=0$). Let $X=t^{l_1+1}|\xi|^{2n}$, ($|\xi|>c>0$), then considering that $t^{l_0+1}|\xi|^{m-1}\leqq X^{(l_0+1)/(l_1+1)}|\xi|^{-\delta_1}$, we have

$$\left|\partial_\xi^\mu\int_0^t\mathcal{L}(\tau,\xi)d\tau\right|\leqq C(1+|\xi|)^{-\mu+1-\delta_1}\cdot\{X^{(l_0+1)/(l_1+1)}+X\},\qquad \mu\geqq 1,$$

which implies immediately (2.6). On the other hand, since $\delta_1\leqq 1$,

$\delta_1\mu > ml$ implies that (2.4) there exists at least one of terms of

$$\partial_\xi^{\mu_i} \int_0^t \mathcal{L}(\tau, \xi) d\tau,$$

which shows (2.7). Q.E.D.

Now we note that $F(x-y, t, s)$ is regular in (x, t) and also in (y, s) is an immediate consequence from (2.5) but it is not very regular in the sense of L. Schwartz. But we can prove that $F(x-y, t, 0)$ is very regular (in some sense). In fact, we have

Lemma 2. $F(x-y, t, 0) \in C^\infty(V)$, $V = \{(x, y, t) \in \boldsymbol{R}_x \times \boldsymbol{R}_y \times (-1, 1)\,;\ |x-y|+|t|>0\}$.

Proof. When $|t|>0$, $F(x-y, t, 0)$ is infinitely differentiable, since $\hat{E}(\xi, t, 0) \in \mathcal{E}^\infty((0,1)\,; \mathcal{S}_\xi)$, $(F=0$ in $t<0)$. Let $l+\beta \leqq N$, (N is an arbitrary fixed positive integer), then we choose α such as $\delta_1\alpha > mN+1$ and let us consider $\partial_t^l \partial_x^\beta x^\alpha F(x, t, 0)$. Then from $\mathcal{F}_x[\partial_t^l \partial_x^\beta x^\alpha F(x, t, 0)] = \partial_t^l (i\xi)^\beta (i\partial_\xi)^\alpha \hat{E}(\xi, t, 0)$, $(t>0)$ and in view of Lemma 1, it holds that $|\partial_t^l \xi^\beta \partial_\xi^\alpha \hat{E}(\xi, t, 0)| \leqq C(1+|\xi|)^{-(1+\varepsilon)}$ for some positive constant ε, and $\partial_t^l \xi^\beta \partial_\xi^\alpha \hat{E} \rightrightarrows 0$ as $t \searrow 0$, which implies that $x^\alpha F(x, t, 0) \in C^N(\boldsymbol{R}_x \times (-1, 1))$. Considering the arbitrariness of N, our assertion follows immediately.

Q.E.D.

Lemma 2 shows that if $X \in \mathcal{E}^0(J_t\,; \mathcal{E}'(I_k))$ and $X \in C^\infty(\Omega_1)$, where $\Omega_1 \subset I_x \times J_t$ then it holds that

(2.8) $$\langle F(x-y, t, 0), X \rangle_{x,t} \in C^\infty(\Omega_1 \cap \{t=0\}).$$

At first we note that $F(x-y, t, 0)$ defines a continuous mapping from $\mathcal{D}(\boldsymbol{R}_y)$ to $\mathcal{E}(\boldsymbol{R}_x \times J_t^+)$, where $J_t^+ = [0, \beta)$. Therefore $\langle F(x-y, t, 0), X \rangle_{x,t}$ is well defined if $X \in \mathcal{E}^0(J_t\,; \mathcal{E}'(I_x))$ and it belongs to $\mathcal{D}'(\boldsymbol{R}_y)$. Let $\Omega_2 \subset \Omega_1 \subset I_x \times J_t$ and let $\alpha(x, t) \in C_0^\infty(\Omega_1)$ and $\alpha \equiv 1$ in a neighborhood of $\bar{\Omega}_2$, then we have that $\langle F(x-y, t, 0), X \rangle_{x,t} = \langle F(x-y, t, 0), \alpha X \rangle_{x,t} + \langle F(x-y, t, 0), (1-\alpha) X \rangle_{x,t}$. Since $\alpha X \in C_0^\infty(I_x \times J_t)$, the first term in the right hand side belongs to $C^\infty(\boldsymbol{R}_y)$. On the other hand if $y \in \Omega_2 \cap \{t=0\}$, $F(x-y, t, 0)$ is infinitely differentiable in (x, y, t) when $(x, t) \in \bar{\Omega}_2^c$, which implies that the second term in the right hand side belongs to $C^\infty(\Omega_2 \cap \{t=0\})$, which shows (2.8).

Proof of Theorem-(i). Let ${}^tPu \in C^\infty(I_x \times J_t)$, $u \in \mathcal{E}^0(J_t\,; \mathcal{D}'(I_x))$. Let $\Omega_1 \subset I_x \times J_t$ and let $\beta(x, t) \in C_0^\infty(I_x \times J_t)$, $\beta \equiv 1$ in a neighborhood of $\bar{\Omega}_1$, then considering that ${}^tP(\beta u) = \beta {}^tPu + X$ where $X \in \mathcal{E}^0(J_t\,; \mathcal{E}'(I_x))$ and $X = 0$ in a neighborhood of $\bar{\Omega}_1$, we have

(2.9) $$(\beta u)(y, 0) = \langle F(x-y, t, 0), \beta {}^tPu \rangle_{x,t} + \langle F(x-y, t, 0), X \rangle_{x,t}.$$

Therefore, in view of the regularity of $F(x-y, t, 0)$ and from (2.8), we have that $(\beta u)(y, 0)$ belongs to $C^\infty(\Omega_1 \cap \{t=0\})$. This completes the proof. Q.E.D.

3. Proof of Theorem-(ii). Without loss of generality, we may assume that $t=t_0>0$ and $a>0$. And also instead of the operator (1.1) we may consider the following operator,

$$P(t; D_x, \partial_t) = \partial_t + iD_x^m + a(t)D_x^{2n}, \qquad a(t) \neq 0. \tag{3.1}$$

In fact, the following transformation of the coordinates derives the operator (3.1) immediately;

$$T = \int_{t_0}^{t} a\tau^{l_0} d\tau, \quad X = x.$$

Hence we shall prove that the operator (3.1) is not hypoelliptic with respect to x on $t=0$. At first, we prepare a necessary condition for the operator to be hypoelliptic with respect to x on $t=0$.

Lemma 3. *Let $P(x, t; D_x, \partial_t)$ be a differential operator with C^∞-coefficients defined in a domain Ω. Then if P is hypoelliptic on $t=0$ with respect to $x (0 \in \Omega)$, it holds that for any positive integer N and compact set K_1 in $\Omega (0 \in K_1)$, there exist a positive constant C, a positive integer M and a compact set K_2 in Ω such that*

$$\begin{aligned} &\sum_{|\alpha| \leqq N} \sup_{K_1 \cap \{t=0\}} |D_x^\alpha u(x, 0)| \\ &\leqq C\left\{\sup_{K_2} |u(x, t)| + \sum_{|\alpha| \leqq M} \sup_{K_2} |D_{x,t}^\alpha Pu|\right\} \qquad \textit{for any } u \in C^\infty(\Omega). \end{aligned} \tag{3.2}$$

Proof. Suppose that P is hypoelliptic on $t=0$ with respect to x. Let $X = \{u \in C^0(\Omega) ; Pu \in C^\infty(\Omega)\}$, then if we introduce the following semi-norms in X, X is a Fréchet space;

$$\sup_K |u(x, t)| + \sum_{|\alpha| \leqq N} \sup_K |D_{x,t}^\alpha Pu|, \qquad 0 \in K \subset \Omega.$$

From the closed graph theorem, the mapping;

$$u(x, t) \to u(x, 0) \in C^\infty(\Omega \cap \{t=0\}), \qquad (u \in X)$$

is continuous, which implies the inequality (3.2). Q.E.D.

Now we consider the equation,

$$Pu = 0. \tag{3.3}$$

Let

$$u_\xi(x, t) = \sum_{j \geqq 0} u_j(x, t) \cdot \exp[-it\xi^m + ix\xi] \cdot \xi^{-j}, \qquad (\xi > c > 0), \tag{3.4}$$

be a formal solution of $Pu=0$, then $u_j(x, t)$ are easily determined. In fact, substituting (3.4) into (3.3) we have

$$\begin{aligned} Pu_\xi = \exp[-it\xi^m + ix\xi] \cdot &\Big\{\sum_{j \geqq 0} (-i\xi^m u_j + \partial_t u_j)\xi^{-j} \\ &+ \sum_{j \geqq 0} (i\xi^m u_j + imD_x u_j \xi^{m-1} + i \sum_{l \geqq 2} C_l^m D_x^l u_j \xi^{m-l})\xi^{-j} \\ &+ \sum_{j \geqq 0} (a(t)\xi^{2n} u_j + a(t) \sum_{l \geqq 1} C_l^{2n} D_x^l u_j \xi^{2n-l})\xi^{-j}\Big\}. \end{aligned}$$

Comparing the coefficients of ξ^k, $(k \leqq m-1)$ we can obtain

$$u_j(x, t) \in C^\infty(R_x \times (-1, 1))$$

step by step, (we choose $u_0(x, t)$ so that $u_0(0, 0) \neq 0$). Now let $u_\xi^{(k)}(x, t)$ be the k-th approximate solution of $Pu=0$, i.e.,

$$u_\xi^{(k)}(x, t) = \sum_{0 \leqq j \leqq k} u_j(x, t) \cdot \exp[-it\xi^m + ix\xi]\xi^{-j}, \tag{3.5}$$

then we have

$$|D_x u_\xi^{(k)}(0, 0)| \geqq c\xi \qquad \text{as } \xi \to \infty \text{ since } u_0(0, 0) \neq 0, \tag{3.6}$$

$$\sup_K |u_\xi^{(k)}(x,t)| \leqq C \qquad \text{as } \xi \to \infty \ (K \subset \Omega), \tag{3.7}$$

$$\sum_{|\alpha| \leqq M} \sup_K |D_{x,t}^\alpha P u_\xi^{(k)}| = O(\xi^{mM-k+m-2}) \qquad \text{as } \xi \to \infty. \tag{3.8}$$

Let (in Lemma 3) $N=1$ and K_1 is an arbitrary compact set including the origin, then if we choose k so that $k \geqq m(M+1)-2$ for any fixed positive integer M, we see that the inequality (3.2) does not hold, which proves the non-hypoellipticity of the operator (3.1). Q.E.D.

Remark. Let us consider

$$P(t; D_x, \partial_t) = \partial_t + it^2 D_x^3 + t^2 D_x^2, \tag{3.9}$$

which does not satisfy the condition (1.2)–(iv). Then we can show that this is not hypoelliptic on $t=0$ with respect to x. In fact, we can construct a formal solution $u_\xi(x,t) = \sum_{j \geq 0} u_j(x) \cdot \exp[-it^3\xi^3/3 + ix\xi]\xi^{-j}$, $(u_0(0) \neq 0)$ of $Pu=0$ by the same way as (3.4). This shows the non-hypoellipticity of P on $t=0$ with respect to x.

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