5. The Determinant of Matrices of Pseudo-differential Operators

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The purpose of this paper is to give a definition of the determinant of matrices of pseudo-differential operators (of finite order) and to establish some of its properties. Let X be a complex manifold, and P^*X (resp. T^*X) be its cotangent projective (resp. vector) bundle. The projection from T^*X-X onto P^*X is denoted by γ.

Our result is the following.

Theorem. *For every matrix* $A(x,D)=(A_{ij}(x,D))_{1\le i,j\le N}$, *whose entries* $A_{ij}(x,D)$ *are pseudo-differential operators defined on an open set* $U\subset P^*X$, *one can canonically associate* $\det A(x,D)$, *which is a homogeneous holomorphic function defined on* $\gamma^{-1}(U)$, *and possesses the following properties*

a) $\det A(x,D)B(x,D)=\det A(x,D)\cdot\det B(x,D)$

b) $\det(A(x,D)\oplus B(x,D))=\det A(x,D)\cdot\det B(x,D)$

c) *if there are integers* m_i *and* n_j *such that order* $A_{ij}(x,D)\le m_i+n_j$ *and* $\det(\sigma_{m_i+n_j}(A(x,D))$ *does not vanish identically, then*

$$\det A(x,D)=\det(\sigma_{m_i+n_j}(A_{i,j})),$$

where $\sigma_{m_i+n_j}(A_{ij})$ *denotes the principal symbol of* A_{ij} (*which is* 0 *if* A_{ij} *is of the order* $\le m_i+n_j-1$). *In particular, our determinant reduces to the concept of the principal symbol, if the size* N *is* 1.

d) $A(x,D)$ *is invertible if and only if* $\det A(x,D)$ *vanishes nowhere.*

e) *if* $P(x,D)$ *is a pseudo-differential operator such that* $[P,A]=0$, *then* $\{\sigma(P),\det A\}=0$.

Corollary. *If* $A(x,D)$ *is a matrix of differential operators, then* $\det A(x,D)$ *is a homogeneous polynomial on the fiber coordinate* ξ.

Corollary is an immediate consequence of Theorem. In fact, by adding an auxiliary parameter t, one can regard $A(x,D)$ as a pseudo-differential operator defined on a (t,x)-space $C\times X$. Therefore, $\det A(x,D)$ is defined all over T^*X, which implies $\det A(x,D)$ is a polynomial on ξ.

In order to prove Theorem, we prepare the following lemma.

Lemma (see [2]). *Let* K *be a* (*not necessarily commutative*) *field,* $K=\bigcup_{m\in Z}K_m$ *be a filtration of* K *satisfying*

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1) *The intersection of all* K_m *is zero,*

2) $K_m \subset K_{m+1}$, *and* $K_1 \neq K_0$,

3) K_m *is closed under addition,*

4) $K_{m_1} K_{m_2} \subset K_{m_1+m_2}$,

5) $[K_{m_1}, K_{m_2}] \subset K_{m_1+m_2-1}$,

6) *If* α *does not belong to* K_m, *then* α^{-1} *belongs to* K_{-1-m}.

Then $k = K_0/K_{-1}$ *is a commutative field and* $L = K_1/K_0$ *is a vector space over* k *of dimension* 1 *and* $K_m/K_{m-1} = L^{\otimes m}$. *The canonical homomorphism from* K_m *to* $L^{\otimes m}$ *is denoted by* σ_m. *Then, there is a map* $\det : M(n : K) \to \oplus_{m \in Z} L^{\otimes m}$ *satisfying*

a) $\det(AB) = \det A \det B$.

b) $\det(A \oplus B) = \det A \det B$.

c) *If there are integers* m_i *and* n_j *such that* $A_{ij} \in K_{m_i+n_j}$ *and* $\det(\sigma_{m_i+n_j}(A_{ij}))$ *is non zero, then* $\det(A_{ij}) = \det(\sigma_{m_i+n_j}(A_{ij}))$.

d) $\det A \neq 0$ *if and only if* A *is invertible.*

e) *If* $\alpha \in K$ *centralizes a matrix* A, *then* $\sigma(\alpha)$ *centralizes* $\det A$.

Since this is a purely algebraic lemma, we omit its proof.

Now, let p be a point in P^*X. Let K be a quotient field of a stalk $\mathcal{P}^f_{X,p}$ of $\mathcal{P}^f_X$ at p. Then the canonical filtration of $\mathcal{P}^f_X$ defined by order induces a filtration of K. Then $k = K_0/K_{-1}$ is a field of germs of meromorphic functions at p, and L is a set of germs of homogeneous meromorphic functions of order 1 at p. Thus, we can define $\det A(x, D)$ as a homogeneous *meromorphic* functions defined on $\gamma^{-1}(U)$.

Proposition. $\det A(x, D)$ *is a holomorphic function.*

Proof. We will prove this by the induction on the size of $A(x, D)$. Levi's theorem says that a meromorphic function is holomorphic if it is holomorphic except on a 2-codimensional analytic set. Therefore, it suffices to prove that $\det A(x, D)$ is holomorphic outside a 2-codimensional set. We may assume $\det(A(x, D))$ is holomorphic except on a non singular hypersurface $f = 0$, and $(\sum \xi_i dx_i)|_{f^{-1}(0)} \neq 0$. By a quantized contact transformation, we can set $f = \xi_1$. Let r_{ij} be a multiplicity of $\sigma(A_{ij})$ at $\{\xi_1 = 0\}$. Set $r = \min(r_{ij})$. We prove the proposition by the induction of r.

Without loss of generality, we may assume $r_{11} = r$, and $\sigma(A_{11})/\xi_1^r$ never vanishes by Levi's theorem.

By Späth's theorem, A_{ij} has the form

$$A_{1j} = A_{11} Q_j + R_j$$

where

$$R_j = \sum_{\nu < r} R_{j,\nu}(x, D') D_1^\nu \quad (\text{where } D' = (D_2, \cdots, D_n)).$$

Therefore

$$A(x,D)=\begin{bmatrix} A_{11}, R_2, \cdots\cdots\cdots R_N \\ A_{21}, A_{22}-A_{21}Q_2, \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{bmatrix}\begin{bmatrix} 1 & Q_2 \cdots Q_N \\ & 1 \\ & & \ddots \\ & & & 1 \end{bmatrix}.$$

Setting the first matrix of the right hand side $\tilde{A}(x,D)$, we have $\det \tilde{A}(x,D)=\det A(x,D)$. If one of R_j is non zero, the multiplicity of some $\sigma(R_j)$ at $\{\xi_1=0\}$ is strictly less than r. Therefore the hypothesis of the induction implies $\det \tilde{A}(x,D)=\det A(x,D)$ is holomorphic. If all R_j are zero, $\det \tilde{A}$ is the product of $\sigma(A_{11})$ and the determinant of an $(N-1)\times(N-1)$ matrix of pseudo-differential operators. In this case, also, the hypothesis of induction on the size again implies that $\det A(x,D)$ is holomorphic. q.e.d.

Since Property (d) is proved in the same argument, we omit its proof.

Example.

$$A(x,D)=\begin{bmatrix} xD+\alpha(x) & D^2+\beta(x)D+\gamma(x) \\ x^2 & xD+\delta(x) \end{bmatrix}$$

In this case,

$$\det A(x,D)=\begin{cases} (\alpha+\beta-1-x\gamma)\delta & \text{if it is not zero} \\ \alpha\beta-2\beta+x\beta'-x^2\delta & \text{if } \alpha+\beta-1-x\gamma=0. \end{cases}$$

In fact,

$$A(x,D)=\begin{bmatrix} 1 & \\ & x^2 \end{bmatrix}\begin{bmatrix} 1 & xD+\alpha \\ & 1 \end{bmatrix}\begin{bmatrix} 0 & Q \\ 1 & xD+\beta+2 \end{bmatrix}\begin{bmatrix} 1 & \\ & x^{-2} \end{bmatrix},$$

where $Q=(1-\alpha-\beta+x\gamma)xD+(2+2\gamma x+x^2\delta-\alpha\beta-2\alpha-x\beta')$.

Example. Let $A=(A_{ij})_{1<i,j\leq 2}$. If $A_{21}\neq 0$ $\det A=\sigma(A_{21})\sigma(A_{11}A_{21}^{-1}A_{22}-A_{12})$. If $A_{11}\neq 0$ $\det A=\sigma(A_{11})\sigma(A_{22}-A_{21}A_{11}^{-1}A_{12})$.

References

[1] M. Sato, T. Kawai, and M. Kashiwara: Microfunctions and Pseudo-differential Equations. Lecture note in Math., No. 287, Springer, Berlin-Heidelberg-New York, pp. 265–529 (1973).

[2] E. Artin: Geometric Algebra. Interscience (1957).