2. *Remarks on a Totally Real Submanifold*

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§ 1. **Introduction.** K. Yano and S. Ishihara [8] and J. Erbacher [3] have determined the submanifold M of non-negative sectional curvature in the Euclidean space or in the sphere with constant mean curvature, such that M has a constant scalar curvature and a flat normal connection.

Recently, C. S. Houh [4], S. T. Yau [9], and B. Y. Chen and K. Ogiue [2] have investigated totally real submanifolds in a Kähler manifold with constant holomorphic sectional curvature c.

On the other hand, the authors [5]-[7] studied C-totally real submanifolds in a Sasakian manifold with constant ϕ-holomorphic sectional curvature. In particular, we have dealt with C-totally real submanifolds with flat normal connection in [6].

The purpose of this paper is to obtain the following:

Theorem. *Let M^n be a totally real submanifold in a Kähler manifold $\bar{M}^{2n}$. A necessary and sufficient condition in order that the normal connection is flat is that the submanifold M^n is flat.*

§ 2. **Preliminaries.** Let M^n be a submanifold immersed in a Riemannian manifold $\bar{M}^{n+p}$. Let $\langle\ ,\ \rangle$ be the metric tensor field on $\bar{M}^{n+p}$ as well as the metric tensor induced on M^n. We denote by $\bar{\nabla}$ the covariant differentiation in $\bar{M}^{n+p}$ and ∇ the covariant differentiation in M^n determined by the induced metric on M^n. Let $\mathfrak{X}(\bar{M})$ (resp. $\mathfrak{X}(M)$) be the Lie algebra of vector fields on $\bar{M}$ (resp. M) and $\mathfrak{X}^{\perp}(M)$ the set of all vector fields normal to M^n.

The Gauss-Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y), \tag{2.1}$$

$$\bar{\nabla}_X N = -A^N(X) + D_X N, \quad X, Y \in \mathfrak{X}(M), \quad N \in \mathfrak{X}^{\perp}(M), \tag{2.2}$$

where $\langle B(X, Y), N\rangle = \langle A^N(X), Y\rangle$ and $D_X N$ is the covariant derivative of the normal connection. A and B are called the second fundamental form of M.

The curvature tensors associated with $\bar{\nabla}, \nabla, D$ are defined by the followings respectively:

$$\begin{aligned} \bar{R}(X, Y) &= [\bar{\nabla}_X, \bar{\nabla}_Y] - \bar{\nabla}_{[X,Y]}, \\ R(X, Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \\ R^{\perp}(X, Y) &= [D_X, D_Y] - D_{[X,Y]}. \end{aligned} \tag{2.3}$$

If the curvature tensor $R^{\perp}$ of the normal connection D vanishes, then the normal connection D is said to be flat.

§ 3. **Proof of Theorem.** Let $\bar{M}^{2n}$ be a Kähler manifold with Kähler structure J. A submanifold M^n in $\bar{M}^{2n}$ is called totally real submanifold if each tangent space of M^n is mapped into the normal space by the Kähler structure J.

Now, we shall prove Theorem stated in § 1. Let $E_1, \cdots, E_n$ be orthonormal basis of $\mathfrak{X}(M)$, then by the definition of the totally real submanifold, $\mathfrak{X}^{\perp}(M)$ is spanned by $JE_1, \cdots, JE_n$. Therefore, if $N \in \mathfrak{X}^{\perp}(M)$, then $JN \in \mathfrak{X}(M)$. Since $JY \in \mathfrak{X}^{\perp}(M)$, it follows that

$$\bar{\nabla}_X(JY) = -A^{JY}(X) + D_X(JY), \tag{3.1}$$

by virture of (2.2). On the other hand, we can obtain

$$\bar{\nabla}_X(JY) = J(\bar{\nabla}_X Y) = J(\nabla_X Y) + JB(X, Y). \tag{3.2}$$

Comparing with the normal part of (3.1) and (3.2), we have

$$D_X(JY) = J(\nabla_X Y). \tag{3.3}$$

Operating D_Z to (3.3) and making use of (3.3), we can get

$$D_Z D_X(JY) = D_Z(J(\nabla_X Y)) = J\nabla_Z \nabla_X Y.$$

Interchanging the vectors X and Z in this equation and taking account of (2.3) and (3.3), it holds that

$$R^{\perp}(Z, X)JY = JR(Z, X)Y.$$

This completes the proof.

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