20. Conductor of Elliptic Curves with Complex Multiplication and Elliptic Curves of Prime Conductor

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(Comm. by Kunihiko KODAIRA, M. J. A., Feb. 12, 1975)

1. In Table I, we give the conductor of all the elliptic curves defined over Q, the rational number field, with complex multiplication with the j-invariants in Q. In Table II, we give all the elliptic curves defined over Q of prime conductor $N \leqq 101$, up to isogeny, under Weil's conjecture for $\Gamma_0(N)$.

2. Let E be an elliptic curve over Q with complex multiplication. Then End $(E) \otimes Q = K$ must be an imaginary quadratic field and End (E) is a subring of R, the ring of integers of K, with finite index. Such a subring is of the form $R_f = Z + fR$, where Z is the ring of rational integers and f is the conductor of R_f. Then End (E) has the class number one and there are 13 such R_f's. Hence there are 13 corresponding elliptic curves and the j-invariants of these curves are well-known ([1]), so we can write explicitly their Weierstrass (not always minimal) models. The conductor of these 13 curves can be calculated as Table I below. As is well-known, the reduction at a prime ($\neq 2, 3$) dividing the conductor N of an elliptic curve with complex multiplication is an additive type, that is to say, $\operatorname{ord}_p N = 2$ if $p \neq 2, 3$, therefore it is sufficient to treat the 2 and 3-factors of N in order to calculate N explicitly. Hence in the last column in Table I, we give only the number $2^{e_2}, 3^{e_3}$, where $N = \prod p^{e_p}$.

Table I

Curve	f	K	model	2,3-factors of N
1	1	$Q[\sqrt{-1}]$	$y^2+x^3+Dx=0$ $\Delta=-2^6D^3$, $j=12^3$ (D: fourth power free)	2^5 if $D\equiv 3$ or $D/4\equiv 1$ 2^6 if $D\equiv 1$ or $D/4\equiv 3$ 2^8 if $2\Vert D$ or $2^3\Vert D$
2	1	$Q[\sqrt{-2}]$	$y^2+x^3+4Dx^2+2D^2x=0$ $\Delta=2^9D^6$, $j=20^3$	2^8
3	1	$Q[\sqrt{-3}]$	$y^2+x^3+D=0$ $\Delta=-2^43^3D^2$, $j=0$ (D: sixth power free)	2^23^2 if i) D: cubic, ii) $D\equiv 3$ and iii) $3\nmid D$ or $3^3\Vert D$ 2^43^2 if i) D: cubic, ii) $D\equiv 1$ and iii) $3\nmid D$ or $3^3\Vert D$

Curve	f	K	model	2,3-factors of N
				$2^6 3^2$ if i) D: cubic, ii) $2^3\|D$ and iii) $3\nmid D$ or $3^3\|D$ $2^2 3^3$ if i) D: non-cubic, ii) $D\equiv 3$ or $D/4\equiv 3$ and iii) $3\nmid D$ or $3^3\|D$ $2^4 3^3$ if i) D: non-cubic, ii) $D\equiv 1$, $D/4\equiv 1$ or $D/16\equiv 1$ and iii) $3\nmid D$ or $3^3\|D$ $2^6 3^3$ if i) D: non-cubic, ii) $2\|D$, $2^3\|D$ or $2^5\|D$ and iii) $3\nmid D$ or $3^3\|D$ $2^2 3^5$ if i) $D\equiv 3$ or $D/4\equiv 3$ and ii) $3\|D$, $3^2\|D$, $3^4\|D$ or $3^5\|D$ $2^4 3^5$ if i) $D\equiv 1$, $D/4\equiv 1$ or $D/16\equiv 1$ and ii) $3\|D$, $3^2\|D$, $3^4\|D$ or $3^5\|D$ $2^6 3^5$ if i) $2\|D$, $2^3\|D$ or $2^5\|D$ and ii) $3\|D$, $3^2\|D$, $3^4\|D$ or $3^5\|D$ 3^3 if i) D: non-cubic, ii) $D/16\equiv 3$ and iii) $3\nmid D$ or $3^3\|D$ 3^5 if i) $D/16\equiv 3$ and ii) $3\|D$, $3^2\|D$, $3^4\|D$ or $3^5\|D$
4	1	$Q[\sqrt{-7}]$	$y^2+x^3+21Dx^2$ $+16\cdot 7D^2x=0$ $\Delta=-2^{12}7^3D^6$, $j=-15^3$	2^4 if $D\equiv 1$ 2^6 if $D\equiv 2$ 1 if $D\equiv 3$
5	1	$Q[\sqrt{-11}]$	$y^2+x^3-2^3 33D^2x$ $-2\cdot 7\cdot 11^2D^3=0$ $\Delta=-2^6 3^6 11^3 D^6$, $j=-2^{15}$	$2^6 3^2$ if either $2\nmid D$, $3\nmid D$ or $D/2\equiv 1$, $3\nmid D$ 2^6 if either $2\nmid D$, $3\mid D$ or $D/6\equiv 3$ 3^2 if $3\nmid D$ and $D/2\equiv 3$ 1 if $D/6\equiv 1$
6	1	$Q[\sqrt{-19}]$	$y^2+x^3-2^3 19D^2x$ $+2\cdot 19^2D^3=0$ $\Delta=-2^6 19^3 D^6$, $j=-2^{15}3^3$	2^6 if $2\nmid D$ or $D/2\equiv 1$ 1 if $D/2\equiv 3$
7	1	$Q[\sqrt{-43}]$	$y^2+x^3-2^4 5\cdot 43D^2x$ $+2\cdot 3\cdot 7\cdot 43^2D^3=0$ $\Delta=-2^6 43^3 D^6$, $j=-2^{18}3^3 5^3$	2^6 if $2\nmid D$ or $D/2\equiv 1$ 1 if $D/2\equiv 3$
8	1	$Q[\sqrt{-67}]$	$y^2+x^3-2^3 5\cdot 11\cdot 67D^2x$ $+2\cdot 7\cdot 31\cdot 67^2D^3=0$ $\Delta=-2^6 67^3 D^6$, $j=-2^{15}3^3 5^3 11^3$	2^6 if $2\nmid D$ or $D/2\equiv 1$ 1 if $D/2\equiv 3$

Curve	f	K	model	2,3-factors of N
9	1	$Q[\sqrt{-163}]$	$y^2+x^3-2^4 5\cdot 23\cdot 29\cdot 163D^2x$ $+2\cdot 7\cdot 11\cdot 19\cdot 127$ $\cdot 163^2 D^3=0$ $\Delta=-2^6 163^3 D^6$, $j=-2^{18}3^3 5^3 23^3 29^3$	2^6 if $2\nmid D$ or $D/2\equiv 1$ 1 if $D/2\equiv 3$
10	2	$Q[\sqrt{-1}]$	$y^2+x^3+6Dx^2+D^2x=0$ $\Delta=2^9 D^6$, $j=66^3$	2^5 if $2\nmid D$ 2^6 if $2\nmid D$
11	2	$Q[\sqrt{-3}]$	$y^2+x^3+6Dx^2-3D^2x=0$ $\Delta=2^8 3^3 D^6$, $j=2^4 3^3 5^3$	$2^2 3^2$ if $D\equiv 3$ $2^4 3^2$ if $D\equiv 1$ $2^6 3^2$ if $D\equiv 2$
12	2	$Q[\sqrt{-7}]$	$y^2+x^3-42Dx^2-7D^2x=0$ $\Delta=2^{12}7^3 D^6$, $j=3^3 5^3 17^3$	2^4 if $D\equiv 1$ 2^6 if $D\equiv 2$ 1 if $D\equiv 3$
13	3	$Q[\sqrt{-3}]$	$y^2+x^3-2^3 3\cdot 5D^2x$ $+2\cdot 11\cdot 23D^3=0$ $\Delta=-2^6 3^5 D^6$, $j=-2^{15}3\cdot 5^3$	$2^4 3^3$ if $D/2\equiv 1$ $2^6 3^3$ if $2\nmid D$ 3^3 if $D/2\equiv 3$

Remarks. All congruences are read by modulo 4. Δ and j stand for the discriminant and j-invariant of E respectively and D is a square free integer (except *Curve* 1 and 3). The type of the additive reductions can be computed (troublesomely in some cases) by transforming the model, if necessary, to one of the Néron's standard forms [4, pp. 144–5]; consequently, the 2 and 3-factors of N are listed easily. In *Curves* except *Curve* 3, 5, 11 and 13, needless to say, the 3-factors of N are 3^2 if $3\mid D$. We have, in particular, $N=2^5, 2^6, 2^8, 3^3, 3^5, 7^2, 11^2, 19^2, 43^2, 67^2$ and 163^2 as the prime-power conductor, moreover, all the elliptic curves of $N=2^5, 2^6, 2^8, 3^3, 3^5$ and 7^2 are in Table I (cf. [5], [2]).

3. We list all the elliptic curves of prime conductor $N=p\leqq 101$, up to isogeny, in Table II below under Weil's conjecture, that is, any elliptic curve is parametrized by modular forms for

Table II

N	minimal model	Δ	j	
37	$y^2+y+x^3-x=0$	37	$2^{12}3^3\Delta^{-1}$	*
	$y^2-4xy+y+x^3=0$	37	$2^{15}5^3\Delta^{-1}$	
43	$y^2+y+x^3-x^2=0$	-43	$2^{12}\Delta^{-1}$	
53	$y^2+xy+y+x^3+x^2+x=0$	-53	$-3^3 5^3\Delta^{-1}$	
61	$y^2+xy+y+x^3-3x^2+2x=0$	-61	$97^3\Delta^{-1}$	
67	$y^2+y+x^3+5x^2-4x+1=0$	-67	$2^{12}37^3\Delta^{-1}$	
73	$y^2+xy+x^3+x^2-x=0$	73	$3^3 19^3\Delta^{-1}$	*
79	$y^2+xy+y+x^3-x^2-x=0$	79	$97^3\Delta^{-1}$	
83	$y^2+3xy-y+x^3+x^2=0$	-83	$-47^3\Delta^{-1}$	
89	$y^2+xy+y+x^3-x^2=0$	-89	$7^6\Delta^{-1}$	
	$y^2+xy+x^3-x^2-x=0$	89	$73^3\Delta^{-1}$	*
101	$y^3+y+x^3+2x^2=0$	101	$2^{18}\Delta^{-1}$	

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z})\,;\, c\equiv 0 \pmod N\right\}.$$

From Wada's Table [6] of the characteristic polynomials of Hecke operators, we obtain N's such that no elliptic curve has small prime conductor N under above conjecture. Since the curves of prime conductor N such that the Jacobian variety with respect to $\Gamma_0(N)$ has dimension one, i.e. $N=11, 17, 19$ are well known, we may restrict to N's of dimension $\geqq 2$.

Remarks. $*$ in the last column means that the curve has a rational point of finite order, so their isogenous curves may be, easily found (cf. [3], [2]). On the other hand, for many curves of small prime conductor, Setzer in his thesis has shown the truth of Weil's conjecture. Details in this section will appear elsewhere.

References

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