36. *Groups which Act Freely on Manifolds*

By Minoru NAKAOKA
Department of Mathematics, Osaka University

(Comm. by Kenjiro SHODA, M. J. A., March 12, 1975)

1. Introduction. This paper is concerned with groups which act freely on closed manifolds.[1] Two theorems will be proved as application of theorems in [6].

For any odd integer r, let $P''(48r)$ denote the group with generators X, Y, Z, A and relations

$$X^2=Y^2=Z^2=(XY)^2, \quad A^{3r}=1,$$
$$ZXZ^{-1}=YX, ZYZ^{-1}=Y^{-1}, \quad AXA^{-1}=Y,$$
$$AYA^{-1}=XY, \quad ZAZ^{-1}=A^{-1}.$$

J. Milnor [5] asks if the group $P''(48r)$ can act freely on the 3-sphere. We shall prove

Theorem 1. *If $r>1$, the group $P''(48r)$ can not act freely on any closed manifold M having the* mod 2 *homology of the $(8t+3)$-sphere $(t\geqq 0)$.*

We note that the assertion of Theorem 1 is stated in Corollary 4.17 of [4] whose proof is not correct if r is a power of 3. (See also [6].)

F.B. Fuller [3] proves the following: Let X be a compact polyhedron such that the Euler characteristic is not zero, and let $h: X\to X$ be a homeomorphism. Then the iterate h^i for some $i\geqq 1$ has a fixed point. This shows that if G is a group acting freely on X then any element of G has finite order. By proving a theorem similar to the Fuller theorem, we shall show

Theorem 2. *Let M be a $(2n+1)$-dimensional closed manifold such that the* mod 2 *semichracteristic $\hat{\chi}(M; Z_2)$ is not zero, and let G be a group acting freely on M. Then, for any $T\in G$ of order* 2 *and for any $S\in G$, the commutator $[S, T]$ has finite order.*

2. Proof of Theorem 1. It follows that the subgroup in $P''(48r)$ generated by $\{X, Y\}$ is the quaternion group $Q(8)$ of order 8 and it is a normal subgroup. We see also that the quotient group $P''(48r)/Q(8)$ is generated by the coset $T=[Z]$ and $S=[A]$ with relations $T^2=(TS)^2=S^{3r}=1$, and hence it is the dihedral group $D(6r)$ of order $6r$.

Suppose we have a free action of $P''(48r)$ on M. Let $N=M/Q(8)$ denote the quotient manifold of M under the action of $Q(8)$. Then there is a natural free action of $D(6r)$ on N. Since the homology group

---
1) In this paper we work in the topological category.

$H_q(M; Z_2)$ is trivial if $0<q<8t+3$, it follows that $H_q(N; Z_2)$ is isomorphic with the homology group $H_q(Q(8); Z_2)$ of the group $Q(8)$ if $q<8t+3$. Therefore it holds that

$$H_q(N; Z_2)=\begin{cases} Z_2 & \text{if } q\equiv 0 \text{ or } 3 \bmod 4,\\ Z_2\oplus Z_2 & \text{if } q\equiv 1 \text{ or } 2 \bmod 4\end{cases}$$

if $0\leqq q\leqq 8t+3$ (see [2], p. 254). Thus the dimension of the vector space $H_*(N; Z_2)$ is 2 modulo 4. Under the isomorphism $H_q(N; Z_2) \cong H_q(Q(8); Z_2)$ $(0\leqq q\leqq 8t+3)$, the homomorphism $S_*: H_q(N; Z_2) \to H_q(N; Z_2)$ corresponds to the homomorphism $\sigma_*: H_q(Q(8); Z_2) \to H_q(Q(8); Z_2)$, where $\sigma: Q(8)\to Q(8)$ is a homomorphism given by $\sigma(U)=AUA^{-1}$ $(U\in Q(8))$. Since $A^3XA^{-3}=X$ and $A^3YA^{-3}=Y$, we have $S_*^3=\mathrm{id}$. Thus it follows from iii) of Theorem (6.1) in [6][2] that

$$S^3T=TS^3, \quad \text{i.e.} \quad TS^3T^{-1}=S^3.$$

On the other hand, since $T^2=(TS)^2$ we have $TS^3T^{-1}=S^{-3}$. Consequently $S^6=1$. Since $S^{3r}=1$ with odd r, we get $S^3=1$. This contradicts that $r>1$, and completes the proof.

3. Lefschetz numbers of the iterates of an automorphism. The following proposition is proved in [1] more generally.

Proposition 1. *Let K be a field, and let $E=\{E_q\}_{q\geqq 0}$ be a graded vector space over K such that the dimension of E is finite. Assume that the Euler characteristic $\chi(E)=\sum_q(-1)^q \dim E_q$ taken as an element of K is not zero. Then, for any automorphism $\phi=\{\phi_q\}_{q\geqq 0}: E\to E$ of degree 0, there is a positive integer i such that the Lefschetz number*

$$L(\phi^i)=\sum_q(-1)^q \operatorname{tr}\phi_q^i\in K$$

is not zero.

Proof. We denote by $K[[x]]$ the ring consisting of all the formal power series $s(x)=\sum_{i=0}^{\infty}a_ix^i$ $(a_i\in K)$. For an invertible element $s(x)\in K[[x]]$, let $D(s(x))\in K[[x]]$ denote the logarithmic derivative $s'(x)s(x)^{-1}$. For an element $s(x)\in K[[x]]$ of the form

$$s(x)=\left(\sum_{i=0}^{n-1}a_ix^i\right)\left(\sum_{i=0}^{n}b_ix^i\right)^{-1} \qquad (b_0, b_n\neq 0),$$

we define the conjugate $s^*(x)\in K[[x]]$ by

$$s^*(x)=\left(\sum_{i=0}^{n-1}a_ix^{n-1-i}\right)\left(\sum_{i=0}^{n}b_ix^{n-i}\right)^{-1}.$$

Let $w_q(x)$ denote the characteristic polynomial of the automorphism $\phi_q: E_q\to E_q$. Since $w_q(x)$ is invertible in $K[[x]]$, we put

$$w(x)=\left(\prod_q w_{2q}(x)\right)\left(\prod_q w_{2q+1}(x)\right)^{-1}\in K[[x]].$$

We put also

---

2) The theorems in [6] are proved for smooth group actions on smooth manifolds. However it can be proved that they hold for topological group actions on topological manifolds.

$$L_\phi(x)=\sum_{i=0}^{\infty} L(\phi^i)x^i \in K[[x]].$$

Then, working on the algebraic closure of K, it can be proved by computation that

(3.1) $L_\phi(x)$ is the conjugate of $D(w(x))$

(see Theorem 1 of [1]). This shows that $L_\phi(x)$ admits a representation of the form

$$L_\phi(x)=u(x)v(x)^{-1},$$

where $u(x)$ and $v(x)$ are relatively prime polynomials with $\deg u(x) < \deg v(x)$ if $u(x)\neq 0$. Since $\chi(E)\neq 0$, we have $\deg v(x)>0$. Therefore $L(\phi^i)\neq 0$ for some $i\geqq 1$, and the proof is completed.

4. $\hat{L}(f, g; K)$. Let K be a fixed field. Let M_1, M_2 be K-oriented closed manifolds having the same dimension m. For continuous maps $f, g: M_1\to M_2$, we consider the induced homomorphism $f^*: H^*(M_2; K)\to H^*(M_1; K)$ and the Gysin homomorphism $g_!: H^*(M_1; K)\to H^*(M_2; K)$ for cohomology. An element $L(f, g; K)\in K$ given by

$$L(f, g; K)=\sum_{q=0}^{m}(-1)^q \operatorname{tr}(g_! f^* \mid H^q(M_2; K))$$

is called the Lefschetz number of (f, g) (see [7]). If $M_1=M_2=M$, the number $L(f, id; K)$ is

$$L(f; K)=\sum_{q=0}^{m}(-1)^q \operatorname{tr}(f^* \mid H^q(M; K)),$$

the usual Lefschetz number of f.

If $m=2n+1$, we consider also an element $\hat{L}(f, g; K)\in K$ given by

$$\hat{L}(f, g; K)=\sum_{q=0}^{n}(-1)^q \operatorname{tr}(g_! f^* \mid H^q(M_2; K)).$$

If $M_1=M_2=M$ we write $\hat{L}(f; K)$ for $\hat{L}(f, id; K)$:

$$\hat{L}(f; K)=\sum_{q=0}^{n}(-1)^q \operatorname{tr}(f^* \mid H^q(M; K)).$$

We note that

$$\hat{L}(id; \boldsymbol{Z}_2)=\sum_{q=0}^{n}(-1)^q \dim H^q(M; \boldsymbol{Z}_2) \bmod 2$$

is the mod 2 semicharacteristic $\hat{\chi}(M; \boldsymbol{Z}_2)$ of M.

It is easily seen that

$$\operatorname{tr}(g_! f^* \mid H^{2n+1-q}(M_2; K))=\operatorname{tr}(f_! g^* \mid H^q(M_2; K)).$$

Therefore the following relation holds:

(4.1) $$\hat{L}(f, g; K)-\hat{L}(g, f; K)=L(f, g; K).$$

In particular, $\hat{L}(f, g; K)=\hat{L}(g, f; K)$ if and only if $L(f, g; K)=0$.

A simple computation gives

Proposition 2. *Let $\{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ and $\{\alpha'_1, \alpha'_2, \cdots, \alpha'_s\}$ be homogeneous bases for the vector spaces $\bigoplus_{q=0}^{n} H^q(M_2; K)$ and $\bigoplus_{q=0}^{n} H^{2n+1-q}(M_2; K)$ such that $\langle \alpha_i\alpha_j, [M_2]\rangle=\delta_{ij}$. Then we have*

$$\hat{L}(f, g; K)=\sum_{i=1}^{s}(-1)^{\deg \alpha_i}<(f^*\alpha_i)(g^*\alpha_i'), [M_1]>.$$

Here $[M_i]$ denotes the fundamental class of M_i.

The following corollaries are immediate.

Corollary 1. *Let M_0 be a K-oriented closed manifold of dimension $2n+1$, and let $h: M_0 \to M_1$ be a continuous map. Then we have*

$$\hat{L}(fh, gh; K)=(\deg h)\hat{L}(f, g; K).$$

Corollary 2. *Let $T_i: M_i \to M_i$ $(i=1, 2)$ be an orientation preserving involution, and $f: M_1 \to M_2$ be a continuous map. Then we have*

$$\hat{L}(fT_1, T_2f; K)=\hat{L}(T_2f, fT_1; K).$$

5. Proof of Theorem 2. For $i=1, 2$, let M_i be a $(2n+1)$-dimensional closed manifold on which a free involution T_i is given. For a continuous map $f: M_1 \to M_2$, the author defined in [6] a number $\hat{\chi}(f) \in \boldsymbol{Z}_2$ called the equivariant Lefschetz number of f. It follows from Proposition 2 and its corollaries that

$$\hat{\chi}(f)=\hat{L}(fT_1, T_2f; \boldsymbol{Z}_2)=\hat{L}(T_2f, fT_1; \boldsymbol{Z}_2)$$

and if f is a homeomorphism

$$\hat{\chi}(f)=\hat{L}(fT_1f^{-1}T_2^{-1}; \boldsymbol{Z}_2)=\hat{L}(T_2fT_1^{-1}f^{-1}; \boldsymbol{Z}_2).$$

Thus, by Theorem 5.3 of [6][3] we have

Proposition 3. *If $f: M_1 \to M_2$ is a continuous map such that $\hat{L}(fT_1, T_2f; \boldsymbol{Z}_2) \neq 0$, the map fT_1 and T_2f has a coincidence. In particular, if $f: M_1 \to M_2$ is a homeomorphism such that $\hat{L}(fT_1f^{-1}T_2^{-1}; \boldsymbol{Z}_2) \neq 0$, the homeomorphism $fT_1f^{-1}T_2^{-1}$ has a fixed point.*

We shall now prove the following theorem from which Theorem 2 follows immediately.

Theorem 3. *Let M be a $(2n+1)$-dimensional closed manifold, and $T: M \to M$ be a free involution. Let $h: M \to M$ be a homeomorphism. Then, if the* mod 2 *semicharacteristic $\hat{\chi}(M; \boldsymbol{Z}_2)$ is not zero, there is a positive integer i such that $(hTh^{-1}T^{-1})^i: M \to M$ has a fixed point.*

Proof. Define a graded vector space $E=\{E_q\}_{q \geqq 0}$ over $\boldsymbol{Z}_2$ by

$$E_q=\begin{cases} H^q(M; \boldsymbol{Z}_2) & \text{if } 0 \leqq q \leqq n, \\ 0 & \text{if } q>n. \end{cases}$$

Put $g=hTh^{-1}T^{-1}: M \to M$. Then $g^*: H^*(M; \boldsymbol{Z}_2) \to H^*(M; \boldsymbol{Z}_2)$ defines an automorphism $\phi: E \to E$ of degree 0. We have $\chi(E)=\hat{\chi}(M; \boldsymbol{Z}_2) \neq 0$. Therefore, in virtue of Proposition 1, there is a positive integer i such that $L(\phi^i)=\hat{L}(g^i; \boldsymbol{Z}_2) \neq 0$. We have $g^i=fTf^{-1}T^{-1}$, where $f=g^{(i-1)/2}h$ for odd i and $f=g^{i/2}T$ for even i. Therefore it follows from Proposition 3 that g^i has a fixed point. This completes the proof.

---

3) See the footnote 2).

References

[1] C. Bowszyc: On the Euler-Poincaré characteristic of a map and the existence of periodic points. Bull. Acad. Polon. Soc., **17**, 367–372 (1969).

[2] H. Cartan and S. Eilenberg: Homological Algebra. Princeton (1956).

[3] F. B. Fuller: The existence of periodic points. Ann. of Math., **57**, 229–230 (1953).

[4] R. Lee: Semicharacteristic classes. Topology, **12**, 183–199 (1973).

[5] J. Milnor: Groups which act on S^n without fixed points. Amer. J. Math., **79**, 623–630 (1957).

[6] M. Nakaoka: Continuous maps of manifolds with involution. I. Osaka J. Math., **11**, 129–145 (1974).

[7] J. W. Vick: Homology Theory. An Introduction to Algebraic Topology. Academic Press (1973).