55. Note on Strongly Regular Rings and P_1-Rings

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Throughout, R $(\neq 0)$ will represent a ring. R is called a *reduced ring*, if R contains no non-zero nilpotent elements. As is well-known, in a reduced ring every idempotent is central and the left annihilator $l(T)$ of an arbitrary subset T of the ring coincides with the right one $r(T)$. Following [4], R is said to be left *s-unital*, if $RI=I$ for every left ideal I of R, or equivalently, if every principal left ideal $(a|$ of R coincides with Ra. Needless to say, every regular ring is left s-unital. A left R-module U is defined to be *p-injective*, if for any $(a|$ and any R-homomorphism $f: (a| \to U$ there exists an element $u \in U$ such that $f(x)=xu$ for all $x \in (a|$ (cf. [5]). If R is a regular ring then every left R-module is p-injective. Conversely, if every $(a|$ is p-injective then R is a regular ring. In fact, the identity map $i: (a| \to (a|$ is induced by the right multiplication of some idempotent contained in $(a|$. If R is a *P_1-ring*, i.e., if $aR=aRa$ for any $a \in R$, then the set N of nilpotent elements coincides with $l(R)$ (cf. [3]). Similarly, if $aR=a^2R$ for any $a \in R$ then $N=l(R)$. While, if $aR \subseteq Ra^2$ for any $a \in R$, then N coincides with $l(R^2)$ (cf. [2]). As to other terminologies used here, we follow [1].

Now, the purpose of this note is to prove the following theorems.

Theorem 1. (a) *The following conditions are equivalent*:

(1) *R is a strongly regular ring.*

(2) *R is a reduced ring such that every $(a|$ is either $l(b)$ with some b or Re with some idempotent e.*

(3) *R is a left s-unital, left duo ring such that every irreducible left R-module is p-injective.*

(4) *R is a left duo ring such that every $(a|$ is p-injective.*

(5) *R is a semi-prime P_1-ring.*

(6) *R is a semi-prime ring such that $aR=a^2R$ for any $a \in R$.*

(7) *R is a semi-prime ring such that $aR \subseteq Ra^2$ for any $a \in R$.*

(b) *The following conditions are equivalent*:

(1) *R is a strongly regular ring with* 1.

(2) *R is a reduced ring such that every $(a|$ is $l(b)$ with some b.*

(3) *R is a left duo ring with* 1 *such that every irreducible left R-module is p-injective.*

(4) *R is a P_1-ring with* 1.

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Theorem 2. *The following conditions are equivalent*:

(1) *R is a direct sum of a strongly regular ring and a zero ring.*

(2) *R is a P_1-ring.*

(3) *$aR \subseteq Ra^2$ for any $a \in R$.*

(4) *$l(R) = r(R)$ and $R/l(R)$ is strongly regular.*

(5) *$R^2 = R^3$, $l(R^2) = r(R^2)$, and $R/l(R^2)$ is strongly regular.*

(6) *$aR = a^2R$ and $Ra = Ra^2$ for any $a \in R$.*

(7) *$R/l(R)$ and $R/r(R)$ are strongly regular.*

Obviously, Theorem 1 contains the principal results of [5], and Theorem 2 improves [2, Theorems 1 and 3].

Proof of Theorem 1. (a) It is easy to see that (1) implies (2)–(7).

(2)$\Rightarrow$(1). If $(a^2| = Re$ with an idempotent e then $(a - ae)^2 = 0$ implies $a = ae \in (a^2|$. While, if $(a^2| = l(b)$ then $a^2b = 0$ implies $aba = 0$, and so $(ab)^2 = 0$. Hence, we have $ab = 0$, which means $a \in l(b) = (a^2|$.

(3)$\Rightarrow$(1). To our end, it suffices to show that $Ra + l(a) = R$ which will prove $Ra^2 = Ra = (a|$. If $Ra + l(a) \neq R$, then by [4, Lemma 1 (a)] there exists a maximal (left) ideal M containing $Ra + l(a)$. We consider here the map $f: Ra \to R/M$ defined by $xa \mapsto x + M (x \in R)$. To be easily seen, f is well-defined and is an R-homomorphism. Since R/M is an irreducible left R-module, there exists some $a \in R$ such that $x + M = xab + M = M$ for all $x \in R$. But, this yields a contradiction $R = M$.

(4)$\Rightarrow$(1). Since R is a regular, left duo ring, it is strongly regular by [1, Theorem].

(5)$\Rightarrow$(1). In any rate, R is a reduced ring. If $a^2 = aa'a = a'a^2$ then $(a - aa')^2 = 0$, and hence $a = aa' = aba$ with some b.

Similarly, (6)$\Rightarrow$(1) and (7)$\Rightarrow$(1).

(b) It suffices to prove that (2) implies (1). In fact, R is strongly regular by (a). We set $(a| = l(b) = Re_1$ and $(b| = Re_2$ with some (orthogonal) idempotents e_1, e_2. Then, $e = e_1 + e_2$ is an idempotent and $(r(e))^2 = (r(e_1) \cap r(e_2))^2 = (r(a) \cap r(b))^2 = (r(a) \cap l(b))^2 = (r(a) \cap (a|)^2 = 0$. Hence, $r(e) = 0$ and e is the identity of R.

In advance of the proof of Theorem 2, we state a couple of lemmas.

Lemma 1. *If $l(R^n) = r(R^n)$ and $\bar{R} = R/l(R^n)$ is strongly regular for a positive integer n, then $R = R^{n+1} \oplus l(R^n)$.*

Proof. First, we claim that if e is an idempotent of R then it is central. In fact, $\bar{R}$ being strongly regular, $ae - ea \in l(R^n)$ for any $a \in R$, so that $ae - eae = (ae - ea)e^n = 0$, and similarly $ea - eae = 0$. Thus, $ae = eae = ea$ for any $a \in R$. The strong regularity of $\bar{R}$ implies also $R = R^{n+1} + l(R^n)$. Now, let $x = \sum x_i^{(1)} x_i^{(2)} \cdots x_i^{(n+1)}$ be an arbitrary element of $R^{n+1} \cap l(R^n)$. Then, by the regularity of $\bar{R}$, $x_i^{(1)} \equiv x_i^{(1)} e \pmod{l(R^n)}$ for some (central) idempotent e. Hence, $(x_i^{(1)} - x_i^{(1)} e) R^n = 0$ and $x = \sum x_i^{(1)} e x_i^{(2)} \cdots x_i^{(n+1)} = \sum x_i^{(1)} x_i^{(2)} \cdots x_i^{(n+1)} e = xe^n = 0$, whence it follows

$R=R^{n+1}\oplus l(R^n)$.

Lemma 2. (a) *If R is a P_1-ring then $l(x)=r(x)$ for any $x \in R$.*

(b) *If $aR \subseteq Ra^2$ for any $a \in R$, then $l(x)=r(x)$ for any $x \in R$.*

Proof. Since (a) is [2, Lemma 1] itself, we shall prove (b) only. We claim first that if $yx \in l(R^2)$ then $yx \in l(R)$. By hypothesis, $yx=ry^2$ for some $r \in R$. Then, $0=yxry=ry^2ry$ implies $(yry)^2=0$, namely, $yry \in l(R^2)$. Accordingly, $(ry)^3=0$ and $ry \in l(R^2)$, which implies $yxR=ry^2R=0$. In particular, if $xy=0$ then $ry \in l(R^2)$ implies $ry \in l(R)$ and $yx=ry^2=0$.

Proof of Theorem 2. Our theorem is an easy combination of Theorem 1 (a), Lemma 1 and Lemma 2.

References

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