53. On a Characterization of L^2-well Posed Mixed Problems for Hyperbolic Equations of Second Order

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§ 1. **Introduction and results.** Let Ω be a domain in an n-dimensional euclidean space R^n with smooth boundary $\partial\Omega$. Let P be a t-strictly hyperbolic operator of second order defined in the cylinder $R^1\times\bar{\Omega}$ and B a boundary operator of first order defined on $R^1\times\partial\Omega$. Furthermore we assume that the boundary $\partial\Omega$ is non-characteristic for P and B and the coefficients of P and B are smooth and constant outside a compact set of $R^1\times\bar{\Omega}$. We then consider the following mixed problem (P, B):

$$P(t, x; D_t, D_x)u(t, x)=f(t, x) \qquad (t, x)\in R^1\times\Omega \quad t>0,$$
$$B(t, x; D_t, D_x)u(t, x)=g(t, x) \qquad (t, x)\in R^1\times\partial\Omega \quad t>0,$$
$$D_t^j u(0, x)=h_j(x) \qquad (j=0, 1) \quad x\in\Omega.$$

Here $D_t=-i(\partial/\partial t)$, $D_k=-i(\partial/\partial x_k)$ $(k=1, \cdots, n)$ and $D_x=(D_1, \cdots, D_n)$.

The aim of this paper is to show the following

Theorem. A mixed problem (P, B) is L^2-well posed if and only if every constant coefficients problem frozen the coefficients at a boundary point is L^2-well posed.

For the L^2-well posedness of mixed problems see [3].

The "only if" part of Theorem is a special case of [2], Theorem 1 which is proved by using the results in [4], [6]. When the coefficients of B are real valued, the author characterized, using the method in [3], L^2-well posed mixed problems with constant coefficients by the inequalities among the coefficients and proved the "if" part of Theorem by energy method ([1]). When the coefficients of B are complex valued, a characterization of L^2-well posed mixed problems with constant coefficients is obtained in the same direction as real case ([8]). In general, a mixed problem is L^2-well posed whenever Lopatinski determinant does not vanish ([5], [10]). Under the assumption of L^2-well posedness, Lopatinski determinant does not vanish in the interior of the most inner normal cone (11]) and also does not vanish for $\operatorname{Im}\tau<0$ where τ is the covariable of t ([4]). When Lopatinski determinant vanishes only on the real points where the roots λ are simple, a mixed problem is L^2-well posed in the case of second order ([2], [9]). Here λ is a root of characteristic polynomial with respect to the covariable of normal direction to $\partial\Omega$. Thus the "if" part of theorem is proved if a priori L^2-estimate holds in a neighbourhood of a real point such that λ is double. In this paper we shall construct a symmetrizer near such a point using the considerations (§ 2) in [5], [9] and the results (§ 3, Lemma 1) in [8].

§ 2. **Preliminaries.** For the sake of simplicity of description we shall prove Theorem in the case when P is d'Alembertian and Ω is the half space $R^n_+=\{x\in R^n; x=(x', x_n), x_n>0\}$. However, our argument is applicable to a general case. We shall consider a problem (P, B):

$$Pu=\Big(-D_t^2+\sum_{j=1}^n D_j^2\Big)u=f \qquad \text{in } R^{n+1}_+,$$
$$Bu=\Big(D_n-\sum_{j=1}^{n-1} b_j(t,x')D_j-c(t,x')D_t\Big)u=g \qquad \text{on } R^n.$$

In order to reduce our problem (P, B) to one for 2×2 system of pseudo-differential operator of first order, put

$$U={}^t(u_1,u_2)={}^t(\Lambda u, D_n u) \qquad \text{for } u\in C_0^\infty(\overline{R^{n+1}_+}),$$

where

$$\Lambda u=(2\pi)^{-n}\int_{R^n} e^{i(\tau t+\sigma x')}\Lambda(\tau,\sigma)\hat{u}(\tau,\sigma,x_n)d\xi d\sigma,$$
$$\hat{u}=\int_{R^n} e^{-i(\tau t+\sigma x')}u(t,x',x_n)dtdx',$$
$$\Lambda(\tau,\sigma)=(|\tau|^2+|\sigma|^2)^{1/2}, \quad \tau=\xi-i\gamma(\gamma\geqq 0), \quad \sigma\in R^{n-1}.$$

Then our problem becomes

$$D_nU-M\Lambda U={}^t(0,f)=F \qquad \text{in } R^{n+1}_+,$$
$$u_2-su_1=g \qquad \text{on } R^n.$$

Here the symbols of pseudo-differential operators M, s with parameters γ and x_n are as follows:

$$M(\tau',\sigma')=\begin{pmatrix}0 & 1\\ \tau'^2-|\sigma'|^2 & 0\end{pmatrix} \quad (\tau'=\tau\Lambda(\tau,\sigma)^{-1}, \sigma'=\sigma\Lambda(\tau,\sigma)^{-1}),$$
$$(1)\qquad s(t,x',\tau',\sigma')=\sum_{j=1}^{n-1} b_j(t,x')\sigma'_j+c(t,x')\tau'.$$

We shall construct a symmetrizer Q in the form ([5]):

$$(2)\qquad Q(t,x,\xi',\sigma',\gamma)=\begin{pmatrix}d_0 & d_1\\ d_1 & d_2\end{pmatrix}+i\gamma'\begin{pmatrix}0 & f\\ -f & 0\end{pmatrix},$$

where a positive constant f and the real symbols $d_j(t,x,\xi',\sigma')$ ($j=0,1,2$) of order zero are determined in § 3. The integration by part gives

$$2\,\mathrm{Im}\,(D_nU-M\Lambda U, QU)_{0,\gamma}$$
$$=\langle U, QU\rangle_{0,\gamma}+\mathrm{Im}\,((^*MQ-QM)U,U)_{0,\gamma}+(\text{lower order term}),$$

where

$$(u,v)_{0,\gamma}=(e^{-\gamma t}u, e^{-\gamma t}v)_{L^2(R^{n+1}_+)} \quad \text{and} \quad \langle u,v\rangle_{0,\gamma}=(e^{-\gamma t}u, e^{-\gamma t}v)_{L^2(R^n)}.$$

If

$$(3)\qquad d_0=d_2(\xi'^2-|\sigma'|^2),$$

then it follows from (1) and (2) that

$$(4)\qquad ({}^*MQ-QM)(t,x,\xi',\sigma',\gamma')=2i\gamma'\begin{pmatrix}2d_1\xi'-f(\xi'-|\sigma'|^2) & d_2\xi'\\ d_2\xi' & f\end{pmatrix}+0(\gamma'^2).$$

On the other hand, using boundary condition $u_2=su_1+g$ we get

$$(5)\qquad \begin{aligned}&\langle U, QU\rangle_{0,\gamma}\\ &=\langle (d_0+2d_1\operatorname{Re}s+d_2|s|^2+2\gamma' f\operatorname{Im}s)u_1, u_1\rangle_{0,\gamma}+R(U,g),\end{aligned}$$

where $|R(U,g)|\leqq C(\langle \Lambda^{-1/2}U\rangle^2_{0,\gamma}+\langle \Lambda^{1/2}g\rangle^2_{0,\gamma})$. Using (3) the symbol of pseudo-differential operator appeared in the first term of the right hand side in (5) can be written by the form $K+\gamma' H$, where $H(t,x',\xi',\sigma',\gamma)$ is of order zero and

$$(6)\qquad \begin{aligned}K(t,x',\xi',\sigma')&=2d_1\operatorname{Re}s_0+d_2(|s_0|^2+(\xi'^2-|\sigma'|^2)),\\ s(t,x',\tau',\sigma')&=s_0(t,x',\xi',\sigma')-ic(t,x')\gamma'.\end{aligned}$$

Now we shall consider a real point stated in Introduction at which Lopatinski determinant vanishes and λ is double. Recall the definition of Lopatinski determinant R for our problem (P,B):

$$R(t,x',\tau',\sigma')=\sqrt{\tau'^2-|\sigma'|^2}-s(t,x',\tau',\sigma'),$$

where $\sqrt{\tau'^2-|\sigma'|^2}$ is a root in λ of $\lambda^2+|\sigma'|^2-\tau'^2=0$ whose imaginary part is positive if $\gamma'=-\operatorname{Im}\tau'>0$. Hence, in this case, such a real point $(t_0,x'_0,0,\xi'_0,\sigma'_0)(\gamma'=0)$ satisfies the relations:

$$(7)\qquad \xi_0'^2=|\sigma'_0|^2 \quad\text{and}\quad s_0(t_0,x'_0,\xi'_0,\sigma'_0)=0.$$

Suppose that the following inequalities hold in a neighbourhood of a real point satisfying (7):

$$(8)\qquad K(t,x',\xi',\sigma)\geqq 0 \quad\text{and}\quad \operatorname{Im}({}^*MQ-QM)(t,x,\xi',\sigma',\gamma')\geqq C\gamma' I.$$

Then a priori L^2-estimate follows from a sharp form of Gårding inequality ([7]) and [9], Lemma 7.2 (the treatment of the term $\langle \gamma' H u_1, u_1\rangle_{0,\gamma}$); that is, for a large constant $\gamma>0$,

$$\gamma^2\|\varphi U\|^2_{0,\gamma}\leqq C(\|F\|^2_{0,\gamma}+\gamma^2\langle \Lambda^{1/2}g\rangle^2_{0,\gamma}+\gamma\|u\|^2_{0,\gamma}),$$

where the support of symbol $\varphi(t,x,\xi',\sigma',\gamma')$ of order zero is contained in a small neighbourhood of a point satisfying (7).

We shall introduce a new variable ζ in a neighbourhood of a point satisfying (7):

$$(9)\qquad \zeta=\tau-|\sigma| \ \text{ if } \xi>0 \text{ or } \zeta=\tau+|\sigma| \text{ if } \xi<0.$$

Hereafter we shall consider the first case, since the argument below is applicable to the second case. Then we have

$$(10)\qquad \begin{aligned}s_0(t,x',\xi',\sigma')&=c(t,x')\,|\sigma'|+\sum_{j=1}^{n-1}b_j(t,x')\sigma'_j+c(t,x')\operatorname{Re}\zeta'\\ &=\alpha(t,x',\sigma')+c(t,x')\operatorname{Re}\zeta',\end{aligned}$$

where the second equality is the definition of α. Putting as in [9], § 7

$$(11)\qquad \begin{aligned}d_1(t,x',\xi',\sigma')&=d_1^0(t,x',\sigma')+d_1^1(t,x',\sigma')\operatorname{Re}\zeta',\\ d_2(t,x',\xi',\sigma')&=d_2^0(t,x',\sigma')\end{aligned}$$

(Remark that, in [9], $d_1^1=0$ and the linear term in $\operatorname{Re}\zeta'$ of d_2 is cosidered), then we can rewrite K as a polynomial in $\operatorname{Re}\zeta'$ of degree 2:

$$K(t,x',\xi',\sigma')=K_0+2K_1\operatorname{Re}\zeta'+K_2(\operatorname{Re}\zeta')^2,$$

where

$$(12)\quad \begin{aligned} K_0(t,x',\sigma') &= 2d_1^0 \operatorname{Re}\alpha + d_2^0 |\alpha|^2, \\ K_1(t,x',\sigma') &= d_1^0 \operatorname{Re} c + d_1^1 \operatorname{Re}\alpha + d_2^0(|\sigma'| + \operatorname{Re}(\bar{c}\alpha)), \\ K_2(t,x',\sigma') &= 2d_1^1 \operatorname{Re} c + d_2^0(1+|c|^2). \end{aligned}$$

§ 3. **A construction of a symmetrizer Q.** In this section we shall show that (8) holds in a small neighbourhood of a point satisfying (7), if

$$(13)\qquad d_1^0 = |\sigma'| + \operatorname{Re}(\bar{c}\alpha), \quad d_1^1 = 1+|c|^2, \quad d_2^0 = -\operatorname{Re} c,$$

and f is a large positive constant. Here we remark that the choice of d_1 and d_2 is a natural extension of one in [1]. It follows from (12) and (13) that

$$(14)\quad \begin{aligned} &K_0 = 2|\sigma'| \operatorname{Re}\alpha + \operatorname{Re} c(\operatorname{Re}\alpha)^2 - \operatorname{Re} c(\operatorname{Im}\alpha)^2 + 2\operatorname{Im} c \operatorname{Re}\alpha \operatorname{Im}\alpha, \\ &K_1 = (1+|c|^2)\operatorname{Re}\alpha, \qquad K_2 = (1+|c|^2)\operatorname{Re} c. \end{aligned}$$

In order to prove that $K \geqq 0$, we need the following lemmas.

Lemma 1 ([8]). *A frozen problem $(P, B)_{(t,x')}$ at a boundary point $(t, x', 0)$ is L^2-well posed if and only if, for every σ, either*

(I) *when* $\operatorname{Re}\alpha = \operatorname{Re}\beta = 0$, $1 + |\sigma|^{-2} \operatorname{Im}\alpha \operatorname{Im}\beta > 0$ *or*

(II) *when* $(\operatorname{Re}\alpha)^2 + (\operatorname{Re}\beta)^2 \neq 0$,

$$A = \begin{pmatrix} 2|\sigma|^{-1}\operatorname{Re}\alpha & |\sigma|^{-2}\operatorname{Im}(\alpha\bar{\beta}) \\ |\sigma|^{-2}\operatorname{Im}(\alpha\bar{\beta}) & 2|\sigma|^{-1}\operatorname{Re}\beta \end{pmatrix} \geqq 0,$$

where α is considered as a fuction in σ and

$$(15)\qquad \beta(t,x',\sigma) = c(t,x')|\sigma| - \sum_{j=1}^{n-1} b_j(t,x')\sigma_j.$$

Hereafter β is considered as a function σ'.

Lemma 2.

$$(K_1)^2 - K_0K_2 = -4^{-1}(1+|c|^2)|\sigma'|^2(\det A).$$

Proof.

$$\begin{aligned} &(K_1)^2 - K_0K_2 \\ &\quad = (1+|c|^2)^2(\operatorname{Re}\alpha)^2 - (1+|c|^2)\operatorname{Re} c\,(2|\sigma'|\operatorname{Re}\alpha + \operatorname{Re} c(\operatorname{Re}\alpha)^2 \\ &\qquad - \operatorname{Re} c(\operatorname{Im}\alpha)^2 + 2\operatorname{Im} c\operatorname{Re}\alpha\operatorname{Im}\alpha) \\ &\quad = -(1+|c|^2)(\operatorname{Re}\alpha(2|\sigma'|\operatorname{Re} c - \operatorname{Re}\alpha) - (\operatorname{Re} c\operatorname{Im}\alpha - \operatorname{Im} c\operatorname{Re}\alpha)^2). \end{aligned}$$

Using the relations $\operatorname{Re}\alpha + \operatorname{Re}\beta = 2|\sigma'|\operatorname{Re} c$ and $2(\operatorname{Re} c\operatorname{Im}\alpha - \operatorname{Im} c\operatorname{Re}\alpha) = \operatorname{Im}(\alpha\bar{\beta})$, we obtain the lemma.

Lemma 3. *If the case* (II) *in Lemma* 1 *is valid, then we have*

$$K_0 \geqq 0 \text{ and if } K_0 = 0,\ K_1 = 0.$$

Proof. Put

$$X = 2^{-1}|\sigma'|^{-2}\operatorname{Im}(\alpha\bar{\beta}), \qquad Y = |\sigma'|^{-1}\operatorname{Re}\left(\sum_{j=1}^{n-1} b_j\sigma'_j\right).$$

Then the case (II) is valid if and only if

$$\operatorname{Re} c > 0 \quad \text{and} \quad X^2 + Y^2 \leqq (\operatorname{Re} c)^2.$$

Furthermore we have

$$(16)\quad |\sigma'|^{-2}K_0 = (\operatorname{Re} c)^{-1}(2\operatorname{Re} c(Y + \operatorname{Re} c) + |c|^2(Y + \operatorname{Re} c)^2 - X^2).$$

In fact, since $\operatorname{Re}\alpha = \operatorname{Re} c + Y$, the left hand side of (16) becomes

$$2(\mathrm{Re}\, c+Y)+\mathrm{Re}\, c(\mathrm{Re}\, c+Y)^2$$
$$+\mathrm{Im}\,\alpha\Big(\mathrm{Re}\, c\,\mathrm{Im}\, c+2Y\,\mathrm{Im}\, c-\mathrm{Re}\, c\,|\sigma'|^{-1}\,\mathrm{Im}\Big(\sum_{j=1}^{n-1} b_j\sigma'_j\Big)\Big).$$

Using the relation

$$X=2^{-1}|\sigma'|^{-2}\,\mathrm{Im}\,(\alpha\bar{\beta})=\mathrm{Re}\, c\,|\sigma'|^{-1}\,\mathrm{Im}\Big(\sum_{j=1}^{n-1} b_j\sigma'_j\Big)-Y\,\mathrm{Im}\, c, \tag{17}$$

the above quantity is equal to

$$2(\mathrm{Re}\, c+Y)+\mathrm{Re}\, c\,(\mathrm{Re}\, c+Y)^2+\mathrm{Im}\,\alpha\,(\mathrm{Im}\, c\,(\mathrm{Re}\, c+Y)-X).$$

Again using (17), $\mathrm{Im}\,\alpha=(\mathrm{Re}\, c)^{-1}(\mathrm{Im}\, c(\mathrm{Re}\, c+Y)+X)$ and hence (16) is proved. By a simple computation, we see that a circle $X^2+Y^2=(\mathrm{Re}\, c)^2$ and a hyperbola $K_0=0$ have only one common point $(X,Y)=(0,-\mathrm{Re}\, c)$ at which two curves tangent each other to second order and a hyperbola K_0 intersects to X-axis at two points $(\pm\mathrm{Re}\, c(2+|c|^2)^{1/2},0)$. Therefore the lemma is proved.

Now we return to the proof of (8). Using (10) and (15), it follows from Lemma 1 that $\mathrm{Re}\, c\geqq 0$. Then we see from (14) that $K_2\geqq 0$. In the case (I) it holds that $\mathrm{Re}\,\alpha=0$ and $\mathrm{Re}\, c=0$. Hence we see from this and (14) that $K=0$. Remark that we do not use the inequality in the case (I). In the case (II) we see from Lemmas 1, 2, 3 that $K\geqq 0$. To prove the second assertion in (8), we remark that $\mathrm{Re}\,\zeta'$, $\mathrm{Re}\,\alpha$ and $\mathrm{Im}\,\alpha$ are small in a neighbourhood of a point satisfying (7). Hence, in such a neighbourhood, that $d_1>0$ follows from (11) and (13). Therefore, it follows from (4) that the second assertion in (8) holds if f be taken large.

References

[1] R. Agemi: On energy inequalities of mixed problems for hyperbolic equations of second order. Jour. Fac. Sci. Hokkaido Univ., Ser. I, **21**, 221–236 (1971).

[2] ——: Remarks on L^2-well posed mixed problems for hyperbolic equations of second order. Hokkaido Math. J., **2**, 214–230 (1973).

[3] R. Agemi and T. Shirota: On necessary and sufficient conditions for L^2-well-posedness of mixed problems for hyperbolic equations. Jour. Fac. Sci. Hokkaido Univ., Ser. I, **21**, 133–151 (1970).

[4] ——: On necessary and sufficient conditions for L^2-well-posedness of mixed problems for hyperbolic equations. II. ibid, **22**, 137–149 (1972).

[5] H. O. Kreiss: Initial boundary value problems for hyperbolic system. Comm. Pure Appl. Math., **19**, 277–298 (1970).

[6] K. Kubota: Remarks on boundary value problems for hyperbolic equations. Hokkaido Math. J., **2**, 202–213 (1973).

[7] P. D. Lax and L. Nirenberg: On stability for difference schemes. A sharp form of Gårding inequality. Comm. Pure Appl. Math., **19**, 473–492 (1966).

[8] S. Miyatake: On mixed problem for hyperbolic equation of second order. R. I. M. S. Kyoto Univ., Surikaisekikenkyusyo Kokyuroku, No. 205, 15–27 (1974) (in Japanese).

[9] T. Ohkubo and T. Shirota: On the structure of L^2-well posed mixed prob-

lems for hyperbolic systems of first order (to appear in Hokkaido Math. J., Vol. 4).
[10] R. Sakamoto: Mixed problems for hyperbolic equations. I. J. Math. Kyoto Univ., **10**, 349–373 (1970).
[11] T. Shirota: On the propagation speed of hyperbolic operator with mixed boundary condition. Jour. Fac. Sci. Hokkaido Univ., **22**, 25–31 (1972).