46. Theory of Tempered Ultrahyperfunctions. II

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We continue our study of tempered ultrahyperfunctions and use the same notations as in our previous note [5]. In this paper, we consider exclusively the 1-dimensional case.

§1. Fourier transformation of distributions with properly convex support. Let $K'=[a,b]$ be a closed interval in $\boldsymbol{R}$. We put

$$(1) \qquad h_{K'}(x)=\sup\{x\xi\,;\,\xi\in[a,b]\}=\begin{cases}bx & \text{for } x\geqslant 0,\\ ax & \text{for } x<0.\end{cases}$$

We denote by $H(\boldsymbol{R};K')$ the space of all C^∞ functions f on $\boldsymbol{R}$ for which there exists a constant $\varepsilon>0$ such that for any integer $p\geqslant 0$, $\exp(h_{K'}(x)+\varepsilon|x|)D^pf(x)$ is bounded in $\boldsymbol{R}$, where $D^p=d^p/dx^p$. $H(\boldsymbol{R};K')$ is the inductive limit of FS spaces. The dual space $H'(\boldsymbol{R};K')$ of $H(\boldsymbol{R};K')$ is a space of distributions of exponential growth ([5]).

Proposition 1. *Let β be a C^∞ function on $\boldsymbol{R}$ such that $0\leqslant\beta(x)\leqslant 1$, $\beta(x)=1$ for $x\geqslant B$ (resp. $x\leqslant -B$) and $\beta(x)=0$ for $x\leqslant -B$ (resp. $x\geqslant B$), with some constant $B>0$. Then $\beta(x)\exp(-ix\zeta)\in H(\boldsymbol{R};K')$ if and only if $\operatorname{Im}\zeta<-b$ (resp. $\operatorname{Im}\zeta>-a$).*

Proof. Remark first

$$(2) \qquad |e^{-ix\zeta}|=e^{x\eta},\ |D^pe^{-ix\zeta}|=|\zeta^p|\,e^{x\eta}, \qquad \text{where } \zeta=\xi+i\eta.$$

Therefore, we have

$$\exp(h_{K'}(x)+\varepsilon|x|)\,|D^pe^{-ix\zeta}|=\begin{cases}|\zeta^p|\exp(b+\varepsilon+\eta)x & \text{for } x>0,\\ |\zeta^p|\exp(a-\varepsilon+\eta)x & \text{for } x<0,\end{cases}$$

from which follows the proposition. q.e.d.

We put

$$(3) \qquad \begin{aligned} H'_{(+)}(\boldsymbol{R};K')&=\{T\in H'(\boldsymbol{R};K')\,;\,\operatorname{supp}T\subset[-A,\infty) \text{ for some } A\geqslant 0\},\\ H'_{(-)}(\boldsymbol{R};K')&=\{T\in H'(\boldsymbol{R};K')\,;\,\operatorname{supp}T\subset(-\infty,A] \text{ for some } A\geqslant 0\},\\ H'_{(0)}(\boldsymbol{R};K')&=\{T\in H'(\boldsymbol{R};K')\,;\,\operatorname{supp}T\subset[-A,A] \text{ for some } A\geqslant 0\}.\end{aligned}$$

These are linear subspaces of $H'(\boldsymbol{R};K')$. We put further

$$(3') \qquad \begin{aligned} H'_{+}(\boldsymbol{R};K')&=\{T\in H'(\boldsymbol{R};K')\,;\,\operatorname{supp}T\subset[0,\infty)\},\\ H'_{-}(\boldsymbol{R};K')&=\{T\in H'(\boldsymbol{R};K')\,;\,\operatorname{supp}T\subset(-\infty,0]\},\\ H'_{0}(\boldsymbol{R};K')&=\{T\in H'(\boldsymbol{R};K')\,;\,\operatorname{supp}T=\{0\}\}.\end{aligned}$$

The spaces $H'_+(\boldsymbol{R};K')$, $H'_-(\boldsymbol{R};K')$ and $H'_0(\boldsymbol{R};K')$ are closed subspaces of the space $H'(\boldsymbol{R};K')$.

Let $T\in H'_{(+)}(\boldsymbol{R};K')$ and $\operatorname{supp}T\subset[-A,\infty)$ (resp. $T\in H'_{(-)}(\boldsymbol{R};K')$ and $\operatorname{supp}T\subset(-\infty,A]$). We choose a C^∞ function β such that $0\leqslant\beta(x)\leqslant 1$, $\beta(x)=1$ for $x\geqslant -A-\delta$ (resp. $x\leqslant A+\delta$) and $\beta(x)=0$ for $x\leqslant -A-2\delta$

(resp. $x \geqslant A+2\delta$) with some $\delta>0$. The function

$$\tilde{T}(\zeta)=(2\pi)^{-1/2}(T_x, \beta(x)e^{-ix\zeta}) \tag{4}$$

is independent of the choice of the function β and is defined for $\zeta \in T((-\infty, -b))$ (resp. for $\zeta \in T((-a, \infty))$. $\tilde{T}$ is, by definition, the Fourier transformation of $T \in H'_{(+)}(\boldsymbol{R}; K')$ or $H'_{(-)}(\boldsymbol{R}; K')$. We will denote also $\mathcal{F}T=\tilde{T}$. If $T \in H'_{(0)}(\boldsymbol{R}; K')=H'_{(+)}(\boldsymbol{R}; K') \cap H'_{(-)}(\boldsymbol{R}; K')$, then

$$\mathcal{F}T(\zeta)=\tilde{T}(\zeta)=(2\pi)^{-1/2}(T_x, e^{-ix\zeta}) \tag{4'}$$

is an entire function of ζ.

For an open set Ω of $\boldsymbol{C}$, $\mathcal{A}_0(\Omega)$ denotes the space of all holomorphic functions ψ on Ω for which there exist for any $\varepsilon>0$ an integer $p \geqslant 0$ and a constant $C \geqslant 0$ such that

$$|\psi(\zeta)| \leqslant C(1+|\zeta^p|) \qquad \text{for } \zeta \in \boldsymbol{C} \backslash (\boldsymbol{C} \backslash \Omega)_\varepsilon, \tag{5}$$

where $(\boldsymbol{C}\backslash\Omega)_\varepsilon$ denotes the ε-neighborhood of $\boldsymbol{C}\backslash\Omega$. By the Liouville theorem, $\mathcal{A}_0(\boldsymbol{C})$ is the space of all polynomials. We can show the following theorem (see Hasumi [1]):

Theorem 1. *Let $K'=[a, b]$. The Fourier transformation defined by (4) or (4′) establishes the following isomorphisms:*

$$\mathcal{F}: H'_0(\boldsymbol{R}; K') \rightarrow \mathcal{A}_0(\boldsymbol{C}), \tag{6}$$

$$\mathcal{F}: H'_+(\boldsymbol{R}; K') \rightarrow \mathcal{A}_0(T((-\infty, -b))), \tag{6'}$$

$$\mathcal{F}: H'_-(\boldsymbol{R}; K') \rightarrow \mathcal{A}_0(T((-a, \infty))). \tag{6''}$$

Proof. (6) is well known. Suppose $T \in H'_+(\boldsymbol{R}; K')$. Then by Theorem 3 of [5], for any $\varepsilon>0$ there exist an integer p and a continuous bounded function F such that

$$T(x)=D^p[\exp(bx+\varepsilon x)F(x)]. \tag{7}$$

Put $T_0=T-D^p[\exp(bx+\varepsilon x)Y(x)F(x)]$, where $Y(x)$ is the Heaviside Y-function. We have

$$\begin{aligned}\tilde{T}(\zeta)&=(2\pi)^{-1/2}(T_x, \beta(x)e^{-ix\zeta})\\ &=2(\pi)^{-1/2}(i\zeta)^p \int_0^\infty F(x)\exp(bx+\varepsilon x)\exp(-ix\zeta)dx+\tilde{T}_0(\zeta)\\ &=(2\pi)^{-1/2}(i\zeta)^p \int_0^\infty F(x)\exp((b+\varepsilon+\eta)x)\exp(-ix\xi)dx+\tilde{T}_0(\zeta).\end{aligned}$$

As $\tilde{T}_0(\zeta)$ is a polynomial, there exist an integer p_0 and a constant $C_0 \geqslant 0$ such that

$$|\tilde{T}(\zeta)| \leqslant C_0(1+|\zeta^{p_0}|) \qquad \text{for } \eta<-b-2\varepsilon. \tag{8}$$

Hence $\mathcal{F}(H'_+(\boldsymbol{R}; K')) \subset \mathcal{A}_0(T((-\infty, -b)))$. Similarly we can show $\mathcal{F}(H'_-(\boldsymbol{R}; K')) \subset \mathcal{A}_0(T((-a, \infty)))$.

Let $\varphi \in \mathfrak{H}(T(-K'))$. There exists a positive number ε_0 such that $\varphi \in \mathfrak{H}(T(-K'_{\varepsilon_0}))$. We have for $\eta \in -K'_{\varepsilon_0}=(-b-\varepsilon_0, -a+\varepsilon_0)$

$$\mathcal{F}\varphi(x)=(2\pi)^{-1/2}\int_{\boldsymbol{R}+i\eta}\varphi(\zeta)e^{-ix\zeta}d\zeta. \tag{9}$$

If β is the function as in (4) and if $-b-\varepsilon_0<\eta<-b$ or $-a<\eta<-a+\varepsilon_0$, the integral

$$\beta(x)\mathcal{F}\varphi(x)=(2\pi)^{-1/2}\int_{R+i\eta}\varphi(\zeta)(\beta(x)e^{-ix\zeta})d\zeta \tag{10}$$

converges in the topology of $H(\boldsymbol{R};K')$. Therefore for $T\in H'_{+}(\boldsymbol{R};K')$ (resp. $T\in H'_{-}(\boldsymbol{R};K')$), we have

$$(T,\mathcal{F}\varphi)=\int_{R+i\eta}\mathcal{F}T(\zeta)\varphi(\zeta)d(\zeta) \tag{11}$$

with $-b-\varepsilon_0<\eta<-b$ (resp. with $-a<\eta<-a+\varepsilon_0$). As the Fourier transformation $\mathcal{F}:\mathfrak{H}(T(-K'))\to H(\boldsymbol{R};K')$ is a topological isomorphism (Theorem 4′ of [5]), the Fourier transformation (6′) and (6″) are injective.

We shall prove the Fourier transformation (6′) is surjective. Suppose $\psi\in\mathcal{A}_0(T((-\infty,-b)))$ is given. Fix $\varphi\in\mathfrak{H}(T(-K'))$ and suppose $\varphi\in\mathfrak{H}(T(-K'_{\varepsilon_0}))$. Because of the Cauchy integral theorem, the integral

$$\int_{R+i\eta}\psi(\zeta)\varphi(\zeta)d\zeta \tag{12}$$

is independent of η satisfying $-b-\varepsilon_0<\eta<-b$. We can define a continuous linear functional on $\mathfrak{H}(T(-K'))$ by assigning (12) to $\varphi\in\mathfrak{H}(T(-K'))$. Hence

$$(S,f)=\int_{R+i\eta}\psi(\zeta)(\overline{\mathcal{F}}f)(\zeta)d\zeta \qquad (-b-\varepsilon_0<\eta<-b) \tag{13}$$

defines a continuous linear functional S on $H(\boldsymbol{R};K')$.

We claim that $\operatorname{supp} S$ is contained in $[0,\infty)$. In fact, by the definition, for any $\varepsilon>0$ there exist an integer p_0 and a constant $C_0\geqslant 0$ such that (8) is valid for $\eta=\operatorname{Im}\zeta<-b-\varepsilon$. If the support of f is compact and contained in $(-\infty,-\delta]$, $\delta>0$, then $\overline{\mathcal{F}}f$ is an entire function and for any integer p there exists a constant $C\geqslant 0$ such that

$$|\zeta^p|\,|\overline{\mathcal{F}}f(\zeta)|\leqslant C\exp(\delta\eta) \qquad \text{for } \eta<0.$$

Thus tending $\eta\to-\infty$ in (13), we get $(S,f)=0$. As $\delta>0$ is arbitrary, this shows $\operatorname{supp} S\subset[0,\infty)$.

By (11) we have

$$\int_{R+i\eta}\mathcal{F}S(\zeta)\varphi(\zeta)d\zeta=(S,\mathcal{F}\varphi)=\int_{R+i\eta}\psi(\zeta)\varphi(\zeta)d\zeta.$$

Hence, putting $\psi_0(\zeta)=\mathcal{F}S(\zeta)-\psi(\zeta)$, we have

$$\int_{R+i\eta}\psi_0(\zeta)\varphi(\zeta)d\zeta=\int_{-\infty}^{\infty}\psi_0(\xi+i\eta)\varphi(\xi+i\eta)d\xi=0 \tag{14}$$

for any $\varphi\in\mathfrak{H}(T(-K'_{\varepsilon_0}))$, $-b-\varepsilon_0<\eta<-b$. Because the restriction of $\mathfrak{H}(T(-K'_{\varepsilon_0}))$ on $\boldsymbol{R}_\xi+i\eta\cong\boldsymbol{R}_\xi(\eta\in-K'_{\varepsilon_0}$ being fixed) forms a dense subspace of $\mathcal{S}(\boldsymbol{R}_\xi)$ and the function $\xi\mapsto\psi_0(\xi+i\eta)$ defines a tempered distribution, (14) shows $\psi_0(\xi+i\eta)=0$ as a distribution of ξ, whence $\psi_0=\mathcal{F}S-\psi=0$. This proves the surjectivity of (6′). We can show similarly the Fourier transformation (6″) is surjective. q.e.d.

In order to describe the Fourier images of $H'_{(+)}(\boldsymbol{R};K')$, $H'_{(-)}(\boldsymbol{R};K')$ and $H'_{(0)}(\boldsymbol{R};K')$, we introduce some notations. For an open set Ω in $\boldsymbol{C}$, $\mathcal{A}_{\exp}(\Omega)$ denotes the space of all holomorphic functions ψ on Ω for which the following estimate is valid with some constant $A\geqslant 0$: for any

$\varepsilon>0$, there exist an integer p and a constant $C\geqslant 0$ such that

$$|\psi(\zeta)|\leqslant C\,(1+|\zeta^p|)\exp(A\,|\mathrm{Im}\,\zeta|)\qquad \text{for } \zeta\in \boldsymbol{C}\backslash(\boldsymbol{C}\backslash\Omega)\,. \tag{15}$$

Theorem 2. *Let $K'=[a,b]$. Then the Fourier transformation $\mathcal{F}$ defined by* (4) *or* (4′) *establishes the following linear isomorphisms:*

$$\mathcal{F}: H'_{(0)}(\boldsymbol{R};K')\to\mathcal{A}_{\exp}(\boldsymbol{C}), \tag{16}$$

$$\mathcal{F}: H'_{(+)}(\boldsymbol{R};K'')\to\mathcal{A}_{\exp}(T((-\infty,-b)))\quad and \tag{16′}$$

$$\mathcal{F}: H'_{(-)}(\boldsymbol{R};K')\to\mathcal{A}_{\exp}(T((-a,\infty))). \tag{16″}$$

Proof. $H'_{(0)}(\boldsymbol{R};K')$ being the space of distributions with compact support, (13) is a linear isomorphism by the Paley-Wiener theorem.

Remark that

$$H'_{(+)}(\boldsymbol{R};K')=\{\tau_A T\,;\,T\in H'_{+}(\boldsymbol{R};K'),\,A\in\boldsymbol{R}\}\quad\text{and}$$

$$\mathcal{A}_{\exp}(T((-\infty,-b)))=\{e^{iA\zeta}\psi(\zeta)\,;\,\psi\in\mathcal{A}_0(T((-\infty,-b))),\,A\in\boldsymbol{R}\},$$

where τ_A is the translation: $(\tau_A T)(x)=T(x-A)$. As we have

$$\mathcal{F}(\tau_A T)(\zeta)=e^{-iA\zeta}(\mathcal{F}T)(\zeta), \tag{17}$$

the isomorphism (16′) results from the isomorphism (6′). (16″) can be similarly shown to be an isomorphism. q.e.d.

§ 2. Fourier transformation of distributions of exponential growth. Proposition 2. *We have the following exact sequences of linear spaces:*

$$\begin{array}{ccccccc} 0\to H'_{(0)}(\boldsymbol{R};K') & \to & H'_{(+)}(\boldsymbol{R};K')\oplus H'_{(-)}(\boldsymbol{R};K') & \to & H'(\boldsymbol{R};K') & \to 0\\ \cup & & \cup & & \| & \\ 0\to H'_{0}(\boldsymbol{R};K') & \to & H'_{+}(\boldsymbol{R};K')\oplus H'_{-}(\boldsymbol{R};K') & \to & H'(\boldsymbol{R};K') & \to 0, \end{array} \tag{18}$$

where $S\in H'_{(0)}(\boldsymbol{R};K')$ goes to $(S,-S)$ and

$$(T_+,T_-)\in H'_{(+)}(\boldsymbol{R};K')\oplus H'_{(-)}(\boldsymbol{R};K')$$

goes to $-T_+ + T_-$.

In fact, by Theorem 3 of [5], we can decompose $T\in H'(\boldsymbol{R};K')$ in the form of $T=-T_+ + T_-$.

By the restriction mapping we consider $\mathcal{A}_{\exp}(\boldsymbol{C})$ as a subspace of $\mathcal{A}_{\exp}(\boldsymbol{C}\backslash T(-K'))$ and $\mathcal{A}_0(\boldsymbol{C})$ as a subspace of $\mathcal{A}_0(\boldsymbol{C}\backslash T(-K'))$. We define the quotient spaces

$$H^1_{T(-K')}(\boldsymbol{C};\mathcal{A}_{\exp})=\mathcal{A}_{\exp}(\boldsymbol{C}\backslash T(-K'))/\mathcal{A}_{\exp}(\boldsymbol{C}), \tag{19}$$

$$H^1_{T(-K')}(\boldsymbol{C};\mathcal{A}_0)=\mathcal{A}_0(\boldsymbol{C}\backslash T(-K'))/\mathcal{A}_0(\boldsymbol{C}). \tag{19′}$$

Then we have the following commutative diagram, each row of which is exact:

$$\begin{array}{ccccccc} 0\to \mathcal{A}_{\exp}(\boldsymbol{C}) & \to & \mathcal{A}_{\exp}(\boldsymbol{C}\backslash T(-K')) & \to & H^1_{T(-K')}(\boldsymbol{C};\mathcal{A}_{\exp}) & \to 0\\ \cup & & \cup & & \uparrow & \\ 0\to \mathcal{A}_0(\boldsymbol{C}) & \to & \mathcal{A}_0(\boldsymbol{C}\backslash T(-K')) & \to & H^1_{T(-K')}(\boldsymbol{C};\mathcal{A}_0) & \to 0. \end{array} \tag{20}$$

Now for $(T_+,T_-)\in H'_{(+)}(\boldsymbol{R};K')\oplus H'_{(-)}(\boldsymbol{R};K')$ we put

$$\Phi(\zeta)=\mathcal{F}(T_+,T_-)(\zeta)=\begin{cases}\mathcal{F}T_-(\zeta) & \text{for } \mathrm{Im}\,\zeta>-a\\ \mathcal{F}T_+(\zeta) & \text{for } \mathrm{Im}\,\zeta<-b.\end{cases}$$

Then by Theorems 1 and 2, the Fourier transformation $\mathcal{F}:(T_+,T_-)\to\Phi$ gives a linear isomorphism of

$$H'_{(+)}(\boldsymbol{R};K')\oplus H'_{(-)}(\boldsymbol{R};K')$$

onto $\mathcal{A}_{\exp}(C\setminus T(-K'))$ and a linear isomorphism of $H'_+(R;K')\oplus H'_-(R;K')$ onto $\mathcal{A}_0(C\setminus T(-K'))$. Therefore the Fourier transformation $\mathcal{F}$ gives the following commutative diagrams:

$$\begin{array}{ccccccccc} 0 \to & H'_{(0)}(R;K') & \to & H'_{(+)}(R;K')\oplus H'_{(-)}(R;K') & \longrightarrow & H'(R;K') & \longrightarrow 0 \\ & \downarrow \mathcal{F} & & \downarrow \mathcal{F} & & \vdots\, \mathcal{F} & \\ 0 \to & \mathcal{A}_{\exp}(C) & \longrightarrow & \mathcal{A}_{\exp}(C\setminus T(-K')) & \longrightarrow & H^1_{T(-K')}(C;\mathcal{A}_{\exp}) & \to 0 \end{array} \tag{21}$$

and

$$\begin{array}{ccccccccc} 0 \to & H'_0(R;K') & \to & H'_+(R;K')\oplus H'_-(R;K') & \longrightarrow & H'(R;K') & \longrightarrow 0 \\ & \downarrow \mathcal{F} & & \downarrow \mathcal{F} & & \vdots\, \mathcal{F} & \\ 0 \to & \mathcal{A}_0(C) & \longrightarrow & \mathcal{A}_0(C\setminus T(-K')) & \longrightarrow & H^1_{T(-K')}(C;\mathcal{A}_0) & \longrightarrow 0. \end{array} \tag{21'}$$

Theorem 3. *The Fourier transformation $\mathcal{F}$ gives the linear isomorphism*

$$\mathcal{F}: H'(R;K') \to H^1_{T(-K')}(C;\mathcal{A}_{\exp}) \tag{22}$$

and a topological linear isomorphism

$$\mathcal{F}: H'(R;K') \to H^1_{T(-K')}(C;\mathcal{A}_0) \tag{22'}$$

so that the diagrams (21) *and* (21′) *become commutative.*

Corollary. *The canonical mapping*

$$H^1_{T(-K')}(C;\mathcal{A}_0) \to H^1_{T(-K')}(C;\mathcal{A}_{\exp}) \tag{23}$$

is a linear isomorphism.

The Fourier transformations $\mathcal{F}$ (22) and (22′) can be defined more concretely: For $T\in H'(R;K')$, we choose $T_+\in H'_{(+)}(R;K')$ and $T_-\in H'_{(-)}(R;K')$ such that $T=-T_+ + T_-$. We define

$$\Phi(\zeta)\in \mathcal{A}_{\exp}(C\setminus T(-K'))$$

putting

$$\Phi(\zeta)=\begin{cases} \mathcal{F}T_-(\zeta) & \text{for } \operatorname{Im}\zeta > -a \\ \mathcal{F}T_+(\zeta) & \text{for } \operatorname{Im}\zeta < -b. \end{cases}$$

The function Φ depends on the choice of (T_+, T_-). If $T=-T_+ + T_- = -T'_+ + T'_-$, then $T_+ - T'_+ = T_- - T'_- = S\in H'_{(0)}(R;K')$. Therefore the class $[\Phi]$ of Φ modulo $\mathcal{A}_{\exp}(C)$ is well defined by $T\in H'(R;K')$. By the definition, we have $\mathcal{F}T=[\Phi]$.

Remark. For $\psi\in\mathcal{A}_{\exp}(T(-\infty,-b))$ (resp. $\psi\in\mathcal{A}_{\exp}(T(-a,\infty))$), we put

$$\psi_0(\zeta)=\begin{cases} 0 \quad (\text{resp. } \psi(\zeta)) & \text{for } \operatorname{Im}\zeta > -a \\ -\psi(\zeta) \quad (\text{resp. } 0) & \text{for } \operatorname{Im}\zeta < -b. \end{cases}$$

$[\psi_0]$ denotes the class of ψ_0 modulo $\mathcal{A}_{\exp}(C)$. Then the mapping $\psi \mapsto [\psi_0]$ is injective. We will consider by this mapping

$$\mathcal{A}_{\exp}(T(-\infty,-b)) \subset H^1_{T(-K')}(C;\mathcal{A}_{\exp})$$

and

$$\mathcal{A}_{\exp}(T(-a,\infty)) \subset H^1_{T(-K')}(C;\mathcal{A}_{\exp})$$

By this convention the two definitions of $\mathcal{F}T$ for $T\in H'_{(+)}(R;K')$ or $H'_{(-)}(R;K')$ are consistent.

§ 3. Cohomological representation of tempered ultrahyperfunctions. We shall define an inner product of $H^1_{T(-K')}(\boldsymbol{C}; \mathcal{A}_{\exp})$ and $\mathfrak{H}(T(-K'))$. Let $[\Phi] \in H^1_{T(-K')}(\boldsymbol{C}; \mathcal{A}_{\exp})$ and $\varphi \in \mathfrak{H}(T(-K'))$ be given. As Φ belongs to $\mathcal{A}_{\exp}(\boldsymbol{C}\setminus T(-K'))$, there exists, by the definition, $A \geqslant 0$ such that for any $\varepsilon > 0$ there exist an integer p_0 and a constant C such that

$$|\Phi(\zeta)| \leqslant C(1+|\zeta^{p_0}|) \exp (A\,|\mathrm{Im}\,\zeta|) \qquad \text{for } \zeta \in \boldsymbol{C}\setminus T(-K'_{\varepsilon/2}).$$

For the function φ, there exists $\varepsilon_0 > 0$ such that $\varphi \in \mathfrak{H}(T(-K'_{\varepsilon_0}))$. Therefore, the integrals

$$(24) \qquad -\int_{\partial T(-K'_\varepsilon)} \Phi(\zeta)\varphi(\zeta)\,d\zeta = \int_{-\infty}^{\infty} \Phi(\xi+i(-a+\varepsilon))\varphi(\xi+i(-a+\varepsilon))d\xi - \int_{-\infty}^{\infty} \Phi(\xi+i(-b-\varepsilon))\varphi(\xi+i(-b-\varepsilon))d\xi$$

are defined for α sufficiently small positive number ε. They are independent of ε because of the Cauchy integral formula.

If $\Phi \in \mathcal{A}_{\exp}(\boldsymbol{C})$, then the integrals (24) are zero by the Cauchy integral theorem. Hence we may define

$$(25) \qquad \langle [\Phi], \varphi \rangle = -\int_{\partial T(-K'_\varepsilon)} \Phi(\zeta)\varphi(\zeta)\,d\zeta.$$

Theorem 4. *Suppose* $T \in H'(\boldsymbol{R}; K')$ *and* $\varphi \in \mathfrak{H}(T(-K'))$ *be given. Then we have*

$$(26) \qquad \langle \mathcal{F}T, \varphi \rangle = (T, \mathcal{F}\varphi),$$

where the left term is defined by (25) *and the right term is the canonical inner product of* $H'(\boldsymbol{R}; K')$ *and* $H(\boldsymbol{R}; K')$.

In fact, the formula (11) in the proof of Theorem 1 gives (26).

Theorem 5 (*Cohomological representation of* $\mathfrak{H}'(T(-K'))$). *The inner product* (25) *gives the linear isomorphism*

$$(27) \qquad \mathfrak{H}'(T(-K')) \cong H^1_{T(-K')}(\boldsymbol{C}; \mathcal{A}_{\exp}) = H^1_{T(-K')}(\boldsymbol{C}; \mathcal{A}_0).$$

The dual Fourier transformation $\mathcal{F}_d$ *defined in* [5] *coincides with the above defined Fourier transformation via* (27).

In fact, we have by (26) and the definition of $\mathcal{F}_d$,

$$\langle \mathcal{F}T, \varphi \rangle = (T, \mathcal{F}\varphi) = (\mathcal{F}_d T, \varphi).$$

The Fourier transformation $\mathcal{F}: H'(\boldsymbol{R}; K') \to H^1_{T(-K')}(\boldsymbol{C}; \mathcal{A}_0)$ and the dual Fourier transformation $\mathcal{F}_d: H'(\boldsymbol{R}; K') \to \mathfrak{H}'(T(-K'))$ being isomorphisms, we get the theorem.

References

[1]～[4] are the same as in [5].
[5] Morimoto, M.: Theory of tempered ultrahyperfunctions. I. Proc. Japan Acad., **51**, 87–91 (1975).