69. *Analytic Functions in a Neighbourhood of Boundary*

By Zenjiro KURAMOCHI
Department of Mathematics Hokkaido University

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Let R be an end of a Riemann surface with compact relative boundary ∂R. Let $F_i(i=1,2,\cdots)$ be a connected compact set such that $F_i\cap F_j=0: i\neq j$, $\{F_i\}$ clusters nowhere in $R+\partial R$ and $R-F(F=\Sigma F_i)$ is connected. We call $R'=R-F$ a lacunary end. If there exists a determining sequence $\{\mathfrak{V}_n(\mathfrak{p})\}$ of a boundary component $\mathfrak{p}$ of R such that $\inf_{z\in\partial\mathfrak{V}_n(\mathfrak{p})} G(z,p_0)>\varepsilon_0>0, n=1,2,\cdots$ and $\partial\mathfrak{V}_n(\mathfrak{p})$ is a dividing cut, we say F is completely thin at $\mathfrak{p}$, where $G(z,p_0)$ is a Green's function of R'. If there exists an analytic function $w=f(z): z\in R'$ such that the spherical area of $f(R')$ is finite over the w-sphere, we say R' satisfies the condition S. If there exists a non const. $w=f(z)$ such that $C(f(R'))$ (complementary set of $f(R')$ with respect to w-sphere) is a set of positive capacity, we say R' satisfies the condition B. Then we proved

Theorem ([1]). *Let R be an end of a Riemann surface $\in 0_g$. If F is completely thin at $\mathfrak{p}$ and $R'=R-F$ satisfies the condition* S, *then the harmonic dimension* (*the number of minimal points of R over $\mathfrak{p}$*)$<\infty$.

In this note we show the above theorem is valid under the condition B instead of the condition S. Since if the spherical area of $f(R')<\infty$, we can find a neighbourhood $\mathfrak{V}_{n_0}(\mathfrak{p})$ of $\mathfrak{p}$ such that $C(f(\mathfrak{V}_{n_0}(\mathfrak{p})\cap R'))$ is a set of positive capacity, the result which will be proved is an extension of the theorem.

Let $R\notin 0_g$ be a Riemann surface. Let $V(z)$ be a positive harmonic function in $R-F$ such that $V(z)=\infty$ on F, $V(z)$ is singular in $R-F$ and $D(\min(M,V(z)))\leqq M\alpha$ for any $M<\infty$, α is a const., we call $V(z)$ a generalized Green's function (abbreviated by G.G.), where F is a set of capacity zero. Then

Lemma 1. 1) *Let $V(z)$ be a* G.G. *in R. Then there exists a cons. α such that $D(\min(M,V(z)))=M\alpha$ and $\int_{C_M}\frac{\partial}{\partial n}V(z)ds=\alpha: C_M=\{z\in R: V(z)=M\}$ for any $M<\infty$.* 2). *Let $G(z,p_i)(i=1,2,\cdots)$ be a Green's function and $\{p_i\}$ be a sequence such that $G(z,p_i)$ converges to $G(z,\{p_i\})$. Then $G(z,p)$ and $G(z,\{p_i\})$ are* G.G.s *such that*

$$\int_{C_M}\frac{\partial}{\partial n}G(z,p)ds=2\pi \quad\text{and}\quad \int_{C_M}\frac{\partial}{\partial n}G(z,\{p_i\})ds\leqq 2\pi. \tag{1}$$

Let $R'=\{z\in R: G(z,p_0)>\delta\}$ and let $\hat{R}'$ be the symmetric image of R'

with respect to $\partial R'=\{z\in R; G(z,p_0)=\delta\}$. We have a doubled surface $\tilde{R}'$ by identifying $\partial R'$ with its image. Then $\tilde{R}'\in 0_g$. Let $\{p_i\}$ be a divergent sequence such that $G'(z,p_i)$ converges. In this case we say $\{p_i\}$ determines an ideal boundary point $p: G'(z,p)=\lim_i G'(z,p_i)$, where $G'(z,p_i)$ is a Green's function of R'. We denote by $\varDelta(R')$ all ideal boundary point. Then G-Martin's topology is introduced on $\bar{R}'=R'+\varDelta(R')$ with distance as follows:

$$\delta(p_i,p_j)=\sup_{z\in D_0}\left|\frac{G(z,p_i)}{1+G(z,p_i)}-\frac{G(z,p_j)}{1+G(z,p_j)}\right|,$$

where D_0 is a compact disc in R'.
By (1) we define $G'(p,q)$ for p and $q\in\bar{R}'$ by

$$G'(p,q)=\lim_{M=\infty}\frac{1}{2\pi}\int_{\partial V_M(p)}G'(\zeta,q)\frac{\partial}{\partial n}G'(\zeta,p)ds,$$

where $V_M(p)=\{z\in R': G'(z,p)>M\}$. Then $G'(p,q)$ is lower semicontinuous in $\bar{R}'\times\bar{R}'$ and

Lemma ([2]). *Let* $F=\{z\in\varDelta(R'): G'(z,p_0)\geqq\delta\}$. *Then*

$$D(F)=1/\lim_{n=\infty}\inf_{p_i,p_j\in F}\frac{1}{{}_nC_2}\sum_{\substack{i<j\\i=1}}^{n}G'(p_i,p_j)=0\qquad\text{for any }\delta>0.$$

We suppose Martin's topology M-top. is defined on $\bar{R}$ with kernels $K(z,p)$.s. Let $G_\delta=\{z\in R: G(z,p_0)>\delta\}$ and $\bar{G}_\delta(M)$ be its closure with respect to M-top. Then

Lemma 3 ([3]). *Let $V(z)$ be a positive harmonic function in R and* a G.G. *in R. Then*

$$V(z)=\int K(z,p)d\mu(p),$$

where μ is a canonical mass on $\bigcup_{\delta>0}\varDelta_1(M)\cap\bar{G}_\delta(M)$ and $\varDelta_1(M)$ is a set of minimal boundary points of R.

Let Ω be a domain in the w-sphere such that $C\Omega$ is a set of positive capacity. Let $G^w(w,\zeta)$ be a Green's function of Ω. We define $G^w(p,q)$ for p and $q\in\bar{\Omega}$ by $G^w(p,q)=\overline{\lim}_{\substack{\xi\to p\\ \eta\to q}}G^w(\xi,\eta)$. Then $G^w(q,p)=G^w(p,q)$ and $G^w(w,p)$ is upper semicontinuous on $\bar{\Omega}\times\bar{\Omega}$ and

Lemma 4. *Let F be a closed set on $\bar{\Omega}$. If*

$$D(F)=1/\lim_{n=\infty}\inf_{p_i,p_j\in F}\frac{1}{{}_nC_2}\sum_{\substack{i<j\\i=1}}^{n}G^w(p_i,p_j)=0,$$

F is a set of (logarithmic) capacity zero.

Lemma 5 ([3]). *Let $U(w)$ be a potential such that* $U(w)=\int G^w(w,p)d\mu(p)$. *If* $\int d\mu(p)<\infty$ and $U(w)\geqq\alpha G^w(w,s):\alpha>0$, then μ has mass $\geqq\alpha$ at s.

Let R and $\tilde{R}$ be Riemann surfaces $R\subset\tilde{R}\notin 0_g$. We suppose Martin's topologies $\tilde{M}$ and M-top s are defined over $\tilde{R}$ and R with kernels $\tilde{K}(z,p)$ and $K(z,p)$. Let $G(z,p_0)$ be a Green's function of R. Let $\Delta_1(\tilde{M})$ and $\Delta_1(M)$ be sets of minimal boundary points of $\tilde{R}$ and R, $\mathfrak{p}$ be a boundary component of $\tilde{R}$ and $\Gamma(\mathfrak{p})$ be the set points (relative to $\tilde{M}$ or M top. s) lying over $\mathfrak{p}$. Let $F_\delta(\alpha)=\{z: \overline{\lim_{\xi \underset{\alpha}{\to} z}} G(z,p_0)\geqq\delta\}$, where $\alpha=\tilde{M}$ or M. Then

Lemma 6.

$$F_\delta(\tilde{M})\cap\Delta_1(\tilde{M})\cap\Gamma(\mathfrak{p})\approx F_\delta(M)\cap\Delta_1(M)\cap\Gamma(\mathfrak{p}),$$

where $\approx$ means one to one mapping.

Let $R\subset\tilde{R}\notin 0_g$ be Riemann surface. Let $w=f(z): z\in R$ be an analytic function of bounded type. We shall define another Riemann surface R^*. We can find a segment S in R such that there exists a neighbourhood $v(S)$ of S $f(z)$ is univalent in $v(S)$. Let $\mathcal{F}$ be a leaf with projection$=f(R)$. Let $S_{\mathcal{F}}$ be a segment in $\mathcal{F}$ with projection S. We connect S and $S_{\mathcal{F}}$ crosswise. Then we have a Riemann surface $R^*=(\mathcal{F}-S_{\mathcal{F}})+(R-S)+S$ and $R-S\subset R^*$. Put $f(z)=\text{proj. } z$ for $z\in\mathcal{F}-S_{\mathcal{F}}$. Then $f(z)$ is analytic continuation of $f(z)$ into $\mathcal{F}-S_{\mathcal{F}}$ and we can suppose $w=f(z)$ is defined in R^*. So long as we consider the behaviour of $f(z)$ near the boundary of R, we can use R^* instead of R. Let $\partial\mathcal{F}$ be the relative boundary of $\mathcal{F}$ which is clearly$=\partial(f(R))$ in the w-sphere. Let $u(z)$ be a harmonic measure of $\partial\mathcal{F}$ in R^*. Then by $\tilde{R}\notin 0_g$ $u(z)<1$ in R^*. Let $U(w)=\sum u(z_i): z\in R^*, f(z_i)=w$. Then

Lemma 7 ([1]). $U(w)\leqq 1$.

By use of Lemma 7 we have

Theorem 1. *Let $R\subset\tilde{R}\notin 0_g$ be Riemann surfaces and let $w=f(z)$ be an analytic function of bounded type in R. Then*

1) Let $z_i\xrightarrow{M}p\in\bar{R}$ *and* $z_i\in G_\delta=\{z\in R: G(z,p_0)>\delta\}$. *Then $f(z_i)$ $\to$a uniquely determined point denoted by $f(p)$ and there exists a uniquely determined connected piece $\omega(p)$ such that $\omega(p)\ni z_i$ for $i\geqq i(r)$ lying over $|w-f(p)|<r$ for any $r>0$.*

2) *Let $z_i\xrightarrow{\tilde{M}}p\in\Delta_1(\tilde{M}): G(z_i, p_0)>\delta>0$. Then $f(z_i)\to f(p)$ and there exists uniquely determined connected piece $\omega(p)$ such that $z_i\in\omega(p)$ for $i\geqq i(r)$ for any r.*

Let

$$A(\Delta_1(\tilde{M}),\delta)=\{w: w=f(p): p\in\Delta_1(\tilde{M})\cap\bar{G}_\delta(\tilde{M})\},\ A(\Delta_1(M),\delta)$$
$$=\{w: w=f(p): p\in\Delta_1(M)\cap\bar{G}_\delta(M)\}$$

and

$$A(\Delta(M)\cap\bar{G}_\delta(M))=\{w: w=f(p):\in\Delta(M)\cap\bar{G}_\delta(M)\}.$$

Then we have by Lemmas 2, 4 and Theorem 1

Theorem 2. *$A(\Delta_1(\tilde{M}),\delta)\subset A(\Delta_1(M),\delta)\subset A(\Delta(M),\delta)$ and $A(\Delta(M),\delta)$ is a closed set of capacity zero for any $\delta>0$.*

Let $u(p, M)=\overline{\lim}_{z \overrightarrow{M} p} u(z)$ and $u(p, \tilde{M})=\overline{\lim}_{z \overrightarrow{\tilde{M}} p} u(z)$. Then by Lemma 7 we have

Theorem 3. $\sum u(z_i)+\sum u(p_i, \tilde{M})\leqq 1$ and $\sum u(z_i)+\sum u(p_i, M)\leqq 1$, *where* $z_i \in R$ *and* $p_i \in \Delta_1(\tilde{M})$ $f(z_i)=f(p_i)=w$ *and* $z_i \in \tilde{R}$ $p_i \in \Delta_1(M)$.

Let $R\subset\tilde{R}$ be a lacunary end. It is desirable to formulate the behaviour of analytic functions with respect to $\tilde{M}$-top over $\tilde{R}$ not to M-top over R, to discuss the relation between the existence of analytic functions and the structure of $\Delta(\tilde{M})$, the boundary of $\tilde{R}$. Let p_1 and p_2 be points in $\Delta(\tilde{M})$. If there exists a sequence of curves $\{\Gamma_n\}$ $(n=1,2,\cdots)$ with two endpoints z_n^i $(i=1,2)$ such that $z_n^i \xrightarrow{\tilde{M}} p_i$, $\inf_{z\in\Gamma_n} G(z, p_0)>\varepsilon_0>0$ and $\Gamma_n\to$boundary of $\tilde{R}$, we say p_1 and p_2 are chained. Suppose $f(z)$ is bounded type. Then by Theorem 2 we see $f(p_1)=f(p_2)$ for two chained points in $\Delta_1(\tilde{M})$. Suppose R is an end of a Riemann surface and F is completely thin at $\mathfrak{p}$, then we see easily any two points p_1 and p_2 in $\Delta_1(\tilde{M})\cap \varGamma(\mathfrak{p})$ are chained and $f(p_1)=f(p_2)$. On the other hand, we can find a number n_0 such that $\tilde{R}_{n_0}\ni p_0$ and there exists a const. K such that $u(z)\geqq\frac{1}{K}G(z, p_0)$ in $R-\tilde{R}_{n_0}$ and $u(p, \tilde{M})\geqq\frac{\varepsilon_0}{K}$ for $p\in\Delta_1(\tilde{M})\cap \varGamma(\mathfrak{p})$, where $\{\tilde{R}_n\}$ is an exhaustion of $\tilde{R}$. Let p_i $(i=1,2,\cdots,i_0)$ be a point in $\Delta_1(\tilde{M})\cap \varGamma(\mathfrak{p})$. Then $f(p_1)=f(p_2)=\cdots$ and $u(p_i)\geqq\frac{\varepsilon_0}{K}$. By Theorem 3 $\sum u(p_i, M)\leqq 1$. Hence $i_0\leqq\frac{K}{\varepsilon_0}$. Thus we have following

Theorem 4. *Let* $\tilde{R}$ *be an end and* F *be completely thin at* $\mathfrak{p}$. *If there exists an analytic function* $w=f(z)$ *of bounded type in* $\tilde{R}-F$, *then* $\Delta_1(\tilde{M})\cap \varGamma(\mathfrak{p})$ *consists of at most a finite number of points.*

Let $R=\tilde{R}-F$ be lacunary end. Suppose $\Delta_1(\tilde{M})\cap\overline{G}_\delta(\tilde{M})\cap \varGamma(\mathfrak{p})=\Delta_1(M)\cap\overline{G}_{\delta'}(M)\cap \varGamma(\mathfrak{p})$ for any $\delta'<\delta$. Then we have by Lemma 5 we can find a number $\delta_0>0$ such that

$$\Delta_1(M)\cap\overline{G}_{\delta_0}(M)\cap \varGamma(\mathfrak{p})=\Delta_1(M)\cap\overline{G}_{\delta''}(M)\cap \varGamma(\mathfrak{p}) \text{ for any } \delta''<\delta_0$$

and

$$\{w=f(p): p\in\Delta_1(\tilde{M})\cap\overline{G}_\delta(\tilde{M})\}=\{w=f(P): p\in\Delta_1(M)\cap\overline{G}_{\delta_0}(M)\cap \varGamma(\mathfrak{p})\}.$$

Let $\{z_i\}$ be a sequence in R such that $G(z_i, p_0)>\varepsilon_0>0$ and $z_i\to\mathfrak{p}$. Then we can find a subsequence $\{z_i'\}$ of $\{z_i\}$ such that $z_i'\xrightarrow{M}p\in\Delta(M)\cap\overline{G}_{\varepsilon_0}(M)\cap \varGamma(\mathfrak{p})$, whence $f(z_i)\to f(p)$. Now by $G(z_i, p_0)>\varepsilon_0$, $K(z, p)$ is a G.G and by Lemma 3

$$K(z, p)=\int K(z, q)d\mu(q),$$

where μ is a canonical mass on $\Delta_1(M)\cap\overline{G}_{\delta_0}(M)\cap \varGamma(\mathfrak{p})$. Let $G^w(w, w')$ be a Green's function of $f(R)$. Then by $G(z, z_i)\leqq G^w(f(z), f(z_i))$ we have

$$K(z, q) \leqq \frac{G^w(f(z), f(q))}{\delta_0} \qquad \text{for } q \in \Delta_1(M) \cap \overline{G}_{\varepsilon_0}(M),$$

whence

$$K(z, p) \leqq \frac{1}{\delta_0} \int G^w(f(z), f(q)) d\mu(q) < \infty \qquad \text{by } \int d\mu(q) \leqq 1.$$

Now the mapping $w = f(p) : p \in \Delta(M) \cap \overline{G}_{\delta_0}(M)$ is continuous. There exists a mass ν on

$$A = \{w : w = f(q) : q \in \Delta_1(M) \cap \overline{G}_{\delta_0}(M) \cap V(\mathfrak{p})\}$$

such that

$$\int G^w(f(z), f(q)) d\mu(q) = \int G^w(f(z), t) d\nu(t).$$

Let $E^*K(z, p)$ the lower envelope of superharmonic functions in $f(R)$ larger than $K(z, p)$. Then

$$E^*K(z, p) = aG^w(w, f(p)) \leqq \int G^w(w, t) d\nu(t).$$

This means $f(p) \in A$. Hence we have

Theorem 5. *Let $R = \tilde{R} - F$ be a lacunary end. If there exists a const. such that*

$$\Delta_1(\tilde{M}) \cap \overline{G}_\delta(\tilde{M}) \cap V(\mathfrak{p}) = \Delta_1(\tilde{M}) \cap \overline{G}_{\delta'}(M) \cap V(\mathfrak{p}) \qquad \text{for } \delta' \leqq \delta.$$

Let $w = f(z) : z \in R$ be an analytic function of bounded type. Then

$$\bigcup_{\varepsilon>0} \bigcap_n f(\overline{G_\varepsilon \cap \mathfrak{B}_n(\mathfrak{p})}) = \{w = f(p) : p \in \Delta_1(\tilde{M}) \cap \overline{G}_\delta(\tilde{M}) \cap V(\mathfrak{p})\},$$

where $G_\varepsilon = \{z \in R : G(z, p_0) > \varepsilon\}$ and $\{\mathfrak{B}_n(\mathfrak{p})\}$ is a determining sequence of $\mathfrak{p}$ in $\tilde{R}$.

Applying this theorem to the case F is completely thin at $\mathfrak{p}$, then

$$\bigcup_{\varepsilon>0} \bigcap_n \overline{f(G_\varepsilon \cap \mathfrak{B}_n(\mathfrak{p}))} = f(p_1) = f(p_2), \cdots = f(p_{i_0}).$$

References

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