68. A Note on Isolated Singularity. I

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0. Introduction. This note attempts to generalize the author's earlier result [6] to higher codimensional case, seeking for more profound base of the study. The remarkable feature is the introduction of the condition (L) which provides a reasonable class of isolated singularities including that of complete intersections; in fact almost all important properties are consequences from this condition.

1. Condition L. Let (X, x) be an isolated singularity, namely, a pair of (complex) analytic space X and a point $x \in X$ such that $X \backslash x$ is non-singular.

Definition. We say (X, x) satisfies the condition (L) if and only if $\mathcal{H}_x^q(\Omega_X^p)=0$ for (p, q) such that $p+q<\dim X$, where Ω_X^p denote the sheaves of analytic p-forms on X for $p=0, 1, 2, \cdots$.

Let f be an analytic function on X such that $f(x)=0$, $df_z \neq 0$ for any $z \in X \backslash x$. Then $(f^{-1}(0), x)$ is a new isolated singularity, which we shall denote by (Y, y) in the following. (Note $y=x$.) We set as in Brieskorn [2]

$$\Omega_f^p = \Omega_X^p / df \wedge \Omega_X^{p-1}.$$

Now we have

Theorem 1. *Let $n=\dim Y \geqq 2$. Then (X, x) satisfies (L) if and only if (Y, y) satisfies (L) and $\dim \mathcal{H}_y^0(\Omega_Y^n)=\dim \mathcal{H}_y^1(\Omega_Y^n)$.*

Remark. Even in case $n=1$ the condition (L) for (X, x) implies the condition (L) for (Y, y).

For the proof of Theorem 1 we have introduced the following new condition

(L′) $\qquad \mathcal{H}_x^q(\Omega_f^p)=0 \qquad$ for (p, q) such that $p+q<\dim X$

showing that this is equivalent to the both statements of the theorem whose equivalence is to be proved.

By Hamm [4] we obtain

Corollary 1. *It (X, x) is a complete intersection of hypersurfaces, then it satisfies (L).*

Consider now the spectral sequence $'E_2^{p,q}=\mathcal{H}_x^p(\mathcal{H}^q(\Omega_X^{\cdot}))$. These E_2-terms are 0 except $'E_2^{p,0}=\mathcal{H}_x^p(\boldsymbol{C})$, $'E_2^{0,q}=H^q(\Omega_{X,x}^{\cdot})$, $q>0$. But it can be shown by Bloom-Herrera [1] that $H^{r-1}(\Omega_{X,x}^{\cdot})={}'E_r^{0,r-1} \xrightarrow{d_r} {}'E_r^{r,0}=\mathcal{H}_x^r(\boldsymbol{C})$ is zero map for every $r>0$. Comparing this with another spectral sequence $''E_1^{p,q}=\mathcal{H}_x^q(\Omega_x^p)$ which converges to the same limit, we obtain

Corollary 2. *If (X,x) satisfies* (L), *then $\mathcal{H}_x^p(C)=H^p(\Omega^\cdot_{X,x})=0$ for $p<\dim X$.*

By similar method we obtain also

Corollary 3. *Let (X,x) satisfy* (L) *and f be as above. Then the sequence*

$$0\longrightarrow\Omega_X^0\xrightarrow{df}\Omega_X^1\xrightarrow{df}\cdots\xrightarrow{df}\Omega_X^{\dim X}$$

is exact, where $\Omega_X^p\xrightarrow{df}\Omega_X^{p+1}$ denotes the exterior multiplication by df.

In case (X,x) is a complete intersection, Corollaries 2 and 3 have already been proved in Greuel [3].

2. Further results. First we introduce some new complexes:

$$'\Omega^\cdot_f=0\longrightarrow\Omega_f^0\xrightarrow{d}\Omega_f^1\xrightarrow{d}\cdots\xrightarrow{d}\Omega_f^n\longrightarrow0$$

$$'\Omega^\cdot_Y=0\longrightarrow\Omega_Y^0\xrightarrow{d}\Omega_Y^1\longrightarrow\cdots\xrightarrow{d}\Omega_Y^n\longrightarrow0$$

$$''\Omega^\cdot_Y=0\longrightarrow\Omega_Y^0\xrightarrow{d}\cdots\xrightarrow{d}\Omega_Y^{n-1}\xrightarrow{d}\Omega_Y^n/\mathcal{H}_y^0(\Omega_Y^n)\longrightarrow0,$$

where $n=\dim Y$. Then we obtain

Theorem 2. *Assume (X,x) satisfies* (L). *Then the following statements hold:*

(i) $H^p('\Omega^\cdot_{f,x})=0=H^p((\iota_*\iota^*\Omega^\cdot_f)_x)\qquad(p<n)$

(ii) *$H^n((\iota_*\iota^*\Omega^\cdot_f)_x)$ is torsion-free $\mathcal{O}_{C,0}$-module*

(iii) *There are the following three exact sequences:*

$$0\longrightarrow H^n('\Omega^\cdot_{f,x})\longrightarrow H^n((\iota_*\iota^*\Omega^\cdot_f)_x)\longrightarrow\mathcal{H}_x^1(\Omega_f^n)\longrightarrow0$$

$$0\longrightarrow H^{n-1}((\iota_*\iota^*\Omega^\cdot_Y)_y)\longrightarrow\mathcal{H}_y^1(\Omega_Y^{n-1})$$
$$\longrightarrow H^n((\iota_*\iota^*\Omega^\cdot_f)_x)\underset{\mathcal{O}_{C,0}}{\otimes}(\mathcal{O}_{C,0}/\mathfrak{m})\longrightarrow H^n((\iota_*\iota^*\Omega^\cdot_Y)_y)\longrightarrow0$$

$$0\longrightarrow H^{n-1}((\iota_*\iota^*\Omega^\cdot_Y)_y)\longrightarrow H^{n-1}(\mathcal{H}_y^1(\Omega^\cdot_Y))\longrightarrow H^n(''\Omega^\cdot_Y)$$
$$\longrightarrow H^n((\iota_*\iota^*\Omega^\cdot_Y)_y)\longrightarrow H^n(\mathcal{H}_y^1(\Omega^\cdot_Y))\longrightarrow0,$$

where ι denotes $X\backslash x\hookrightarrow X$ or $Y\backslash y\hookrightarrow Y$ according to the context, and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{C,0}$.

Remark. From (ii) and (iii) it follows that $H^n('\Omega^\cdot_{f,x})$ is torsion free. But, in case (X,x) is a complete intersection, this is also included in a much more general theorem of [3]. It should be remarked that in the proof of Theorem 2 we have not made use of the Morse theory, nor of the Gauss-Mannin connection.

Now we shall discuss the case (X,x) is smooth; things are nice in such a case as is shown by the following theorem:

Theorem 3. *Let (X,x), (Y,y) be as above and assume that (x,x) is non-singular. Then there are isomorphisms which are canonical in a certain sense:*

$$\mathcal{H}_y^0(\Omega_Y^{n+1})\simeq\mathcal{H}_y^1(\Omega_Y^n)\simeq\cdots\simeq\mathcal{H}_y^{n-1}(\Omega_Y^2)$$
$$\mathcal{H}_y^0(\Omega_Y^n)\simeq\mathcal{H}_y^1(\Omega_Y^{n-1})\simeq\cdots\simeq\mathcal{H}_y^{n-1}(\Omega_Y^1).$$

The dimension of all these cohomology groups are equal. Furthermore *the following conditions are equivalent*:

(i) $H^n(\Omega^{\cdot}_{Y,y})=0$

(ii) $H^n({}''\Omega^{\cdot}_{Y,y})=0$

(iii) $\dim H^{n-1}((\iota_*\iota^*\Omega^{\cdot}_Y)_y)=\dim H^{n-1}((\iota_*\iota^*\Omega^{\cdot}_Y)_y)$.

Remark. By Saito [7], these equivalent conditions are equivalent to the quasi-homogeneity of (Y, y).

Remark. In case $\dim Y\geqq 3$ the following exact sequence holds (provided (X, x) satisfies (L)):

$$0\longrightarrow\mathcal{H}^1_y(\Omega^{n-1}_Y)\longrightarrow\mathcal{H}^2_x(\Omega^{n-1}_f)\longrightarrow\mathcal{H}^2_x(\Omega^{n-1}_f)\longrightarrow\mathcal{H}^2_y(\Omega^{n-1}_Y)\longrightarrow 0.$$

Thus $\dim\mathcal{H}^1_y(\Omega^{n-1}_Y)=\dim\mathcal{H}^2_y(\Omega^{n-1}_Y)=\dim R^1\iota_*\iota^*\Omega^{n-1}_Y$. This fact, combined with Theorems 2, 3, proves all of the author's earlier results [6].

Problem. *Is there an isolated singularity which is not a complete intersection, but satisfies* (L)?

The details will appear elsewhere.

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