64. *Some Proof Theoretic-Properties of Dense Linear Orderings and Countable Well-Orderings*

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In this paper we shall state, without proofs, some proof-theoretic results concerning dense linear orderings and countable well-orderings (Theorems A and A′ below). By using them and some extended forms of relativization theorem in Motohashi [3] and [4], we shall give purely syntactic proofs of Lopez-Escobar's Theorem and Morley's Theorem on undefinability of well-orderings ((i) and (ii) of Theorem 12 in Keisler [1]).

Let L be a first order infinitary logic with countable conjunctions, countable disjunctions and equality ($L_{\omega_1\omega}$ in the sense of H. J. Keisler's Book [1]). We assume that L has at least one binary predicate symbol $<$ but no individual constants nor function symbols. By L_0 we denote the sublogic of L which is obtained from L by deleting all the predicate symbols except $<$. Let DO be the axiom of dense linear orderings without endpoints and WO_α be the axiom of well-orderings of type α, for each countable ordinal α (see Scott [2]). Then clearly DO and WO_α are sentences in L_0. A formula A in L is said to be existential if A is obtained from atomic formulas and their negations by some applications of $\bigwedge$ (countable conjunction), $\bigvee$ (countable disjunction) and $\exists$ (existential quantification).

Our first result is the following

Theorem A. *Suppose that A is an existential formula in L. Then the sentence $DO \to A$ is provable in L if and only if the sentence $WO_\alpha \to A$ is provable in L for some countable ordinal number α.*

In order to obtain a syntactic proof of Lopez-Escobar's Theorem ((i) of Theorem 12 in [1]), we require the following form of relativization theorem which is mentioned in [3] and [4].

Suppose P is an unary predicate symbol which does not appear in L. By $L(P)$, we denote the logic obtained from L by adding P as a new predicate symbol. For each formula A in L, by A^P we denote the formula in $L(P)$, which is obtained from A by relativizing every occurrence of quantifiers in A by P. Using these notations, we can express the relativization theorem in the following required style.

Theorem B. *If A and B are sentences in L and $(\exists v)P(v) \wedge A^P \to B$*

is provable in L(P), then there is an existential sentence C in L such that $A \to C$ and $C \to B$ are provable in L.

By using Theorems A and B, we have the following

Theorem C (Lopez-Escobar). *Suppose that T is a countable set of sentences in L. Then $(\exists v)P(v)$, DO^P, T is consistent if and only if $(\exists v)P(v)$, WO^P_α, T is consistent for every countable ordinal number α.*

Proof. $(\exists v)P(v)$, DO^P, T is inconsistent

$\Longleftrightarrow \vdash (\exists v)P(v) \wedge DO^P \to \neg \wedge T$

$\Longleftrightarrow \vdash DO \to A$ and $\vdash A \to \neg \wedge T$ for some existential sentence A

$\Longleftrightarrow \vdash WO_\alpha \to A$ and $\vdash A \to \neg \wedge T$ for some $\alpha < \omega_1$ and existential sentence A

$\Longleftrightarrow \vdash (\exists v)P(v) \wedge WO^P_\alpha \to \neg \wedge T$ for some $\alpha < \omega_1$

$\Longleftrightarrow (\exists v)P(v)$, WO^P_α, T is inconsistent for some $\alpha < \omega_1$. q.e.d.

On the other hand we require more delicate arguments to obtain a syntactic proof of Morley's Theorem. For each countable admissible set $\mathcal{A}$, let $L_{\mathcal{A}}$ be the sublogic of $L_{\omega_1\omega}$ restricted to $\mathcal{A}$ (cf. [1]). Note the fact that $WO_\alpha \in L_{\mathcal{A}}$ for each α in $\mathcal{A}$. Suppose that A is an existential formula in L and X a finite set of free variables such that every free variable in A belongs to X. We define the degree of existence of A and X (denoted by $d.e.(A, X)$) by the following conditions:

(i) $d.e.(A, X) = 0$ if A is an atomic formula or its negation;

(ii) $d.e.\left(\bigvee_{i \in I} A_i, X\right) = \sup_{i \in I} d.e.(A_i, X)$;

(iii) $d.e.\left(\bigwedge_{i \in I} A_i, X\right) = \left(\sup_{i \in I} d.e.(A_i, X)\right) \cdot (n+1)$, where $n = \overline{\overline{X}}$;

(iv) $d.e.((\exists v)A(v), X) = d.e.(A(y), X \cup \{y\}) + 1$, where y is a free variable which does not belong to X.

Clearly $d.e.(A, X)$ is a countable ordinal number. Furthermore if $A \in \mathcal{A}$, then $d.e.(A, X)$ is an ordinal number in $\mathcal{A}$. Note that $d.e.(A, X) = 0$ for each open formula A. Let $d.e.(A) = d.e.(A, \phi)$ for each existential sentence A in L, where ϕ is the empty set. Then we have the following lemma which is used in the proofs of Theorem A above and Theorem A′ below.

Lemma. *Suppose that A is an existential sentence in L_0. Then the sentence $DO \to A$ is provable in L_0 if and only if the sentence $WO_{d.e.(A)} \to A$ is provable in L_0.*

Also we require the following form of relativization theorem which is remarked by Mr. K. Shirai.

Theorem B′. *If A and B are sentences in $L_{\mathcal{A}}$ and $(\exists v)P(v) \wedge A^P \to B$ is provable in $L(P)_{\mathcal{A}}$, then there is an existential sentence C in $L_{\mathcal{A}}$ such that $A \to C$ and $C \to B$ are provable in $L_{\mathcal{A}}$ and every predicate symbol in C occurrs both in A and in B.*

By using our Lemma and Theorem B′ we have the following

Theorem A′. *Suppose that A is an existential sentence in $L_{\mathcal{A}}$. Then the sentence $DO \rightarrow A$ is provable in $L_{\mathcal{A}}$ if and only if the sentence $WO_\alpha \rightarrow A$ is provable in $L_{\mathcal{A}}$ for some ordinal number α in $\mathcal{A}$.*

By using Theorems A′ and B′ instead of Theorems A and B in the proof of Theorem C, we have following

Theorem C′ (Morley). *Suppose that T is a countable set of sentences in $L_{\mathcal{A}}$ such that T is Σ on $\mathcal{A}$. Then $(\exists v)P(v)$, DO^P, T is consistent if and only if $(\exists v)P(v)$, WO_α^P, T is consistent for every α in $\mathcal{A}$.*

References

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