79. *Fundamental Solutions of Mixed Problems for Hyperbolic Equations with Constant Coefficients*

By Mikio TSUJI
Department of Mathematics, Kyoto Industrial University

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§ 1. **Introduction.** We study the structure of singular supports of fundamental solutions of hyperbolic mixed problems with constant coefficients in a quarter space. Duff published a basic paper on this subject ([2]) in 1964. Although its results are precise, the paper seems to be difficult to understand. Matsumura [4] studied it by means of "Localization theorem" developed by L. Hörmander [3] and Atiyah-Bott-Gårding [1], but he did not treat the analysis of the fundamental solutions at branch points appearing in reflection coefficients. In this paper we give the "Generalized localization theorem", and by this theorem we can explain the presence of lateral waves.

We thank Prof. Matsumura for having communicated us that Wakabayashi is publishing a note on the same subject ([7]). His results are more restrictive than ours. A forthcoming paper will give detailed proofs and more precise results.

§ 2. **Notations and representation of fundamental solutions.** Let $\Omega=\{(t,x,y)\,;\,t>0,\ x>0,\ y\in R^n\}$. We consider the problem

$$(2.1)\quad \begin{cases} P(D_t,D_x,D_y)u=0 & \text{in } \Omega \\ B_j(D_t,D_x,D_y)u=0 & \text{on } \bar{\Omega}\cap\{x=0\},\ j=1,2,\cdots,\mu, \\ (u,D_tu,\cdots,D_t^{m-1}u)=(0,0,\cdots,0,i\delta_{(x-l,y)}) & \text{on } \bar{\Omega}\cap\{t=0\}, \end{cases}$$

where i) $D_t=-i\partial_t$, $D_x=-i\partial_x$, $D_y=-i(\partial_{y_1},\partial_{y_2},\cdots,\partial_{y_n})$, ii) $l>0$, and iii) P and B_j $(j=1,2,\cdots,\mu)$ are homogeneous differential operators of degree m and m_j $(j=1,2,\cdots,\mu)$ with constant coefficients. We assume

(A.I) *P is strictly hyperbolic with respect to t,*

(A.II) *$x=0$ is not characteristic with respect for P,*

(A.III) *The mixed problem* (2.1) *is $\mathcal{E}$-well posed.*

The characterization of $\mathcal{E}$-well posedness was given by Sakamoto [5]. We write the dual coordinates of (t,x,y) by $(\sigma,\xi,\eta)\in R^{n+2}$, and put $\tau=\sigma-i\gamma$ $(\gamma>0)$. From (A.I), there exists no real zero of $P(\tau,\xi,\eta)$ with respect to ξ for $\tau=\sigma-i\gamma$ $(\gamma>0),(\sigma,\eta)\in R^{n+1}$. From (A.III), the number of roots of P with positive imaginary parts is equal to μ. Therefore we can represent P as follows:

$$\begin{aligned} P(\tau,\xi,\eta)&=\text{const}\prod_{j=1}^{\mu}(\xi-\xi_j^+(\tau,\eta))\cdot\prod_{j=1}^{m-\mu}(\xi-\xi_j^-(\tau,\eta)) \\ &=\text{const}\,P_+(\tau,\eta\,;\,\xi)\cdot P_-(\tau,\eta\,;\,\xi) \end{aligned}$$

where Im. $\xi_j^{\pm}(\tau,\eta)\gtrless 0$. Here we define the matrix $L(\tau,\eta)$ by

$$L(\tau,\eta)=\left(\frac{1}{2\pi i}\oint_{\Gamma_+}\frac{B_j(\tau,\xi,\eta)\xi^{k-1}}{P_+(\tau,\eta;\xi)}d\xi\right)_{1\leq j,k\leq\mu}$$

where Γ_+ is a simple closed path containing all $\xi_i^+(\tau,\eta)$ $(i=1,2,\cdots,\mu)$. We put $R(\tau,\eta)=\det L(\tau,\eta)$ and $R_{jk}(\tau,\eta)=(k,j)$-cofactor of $L(\tau,\eta)$. From (A.III), we get $R(\tau,\eta)\neq 0$ for $\tau=\sigma-i\gamma$ $(\gamma>0)$ and $(\sigma,\eta)\in R^{n+1}$. Now we construct the fundamental solution of (2.1). We define $E_0(t,x,y;\ell)$ by

$$E_0(t,x,y;\ell)=\left(\frac{1}{2\pi}\right)^{n+2}\int_{R^{n+2}}\frac{e^{i(t\tau+(x-\ell)\xi+y\eta)}}{P(\tau,\xi,\eta)}d\sigma d\xi d\eta$$

which is the solution of $P(D_t,D_x,D_y)E_0=\delta_{(t,x-\ell,y)}$ in R^{n+2}, i.e., describes the incident propagation of waves due to a point source $\delta_{(t,x-\ell,y)}$. We put $E_1(t,x,y;\ell)=u(t,x,y;\ell)-E_0(t,x,y;\ell)$ where u is a fundamental solution of (2.1). Then E_1 is represented as follows:

$$\text{(2.2)}\quad E_1(t,x,y;\ell)=\sum_{j,k=1}^{\mu}\left(\frac{1}{2\pi}\right)^{n+3}\int_{R^{n+3}}\frac{R_{jk}(\tau,\eta)\xi^{j-1}B_k(\tau,\xi',\eta)}{R(\tau,\eta)P_+(\tau,\eta;\xi)P(\tau,\xi',\eta)}\times e^{i(t\tau+x\xi+y\eta-\ell\xi')}d\sigma d\xi d\eta d\xi'.$$

The location of sing supp E_0 is well known. Therefore we aim to determine the location of sing supp E_1.

§ 3. Localization theorem. Let $P_0=(\sigma_0,\xi_0,\eta_0,\xi_0')$ be any point in R^{n+3}. We try to expand $\exp\{-is(t\sigma_0+x\xi_0+y\eta_0-\ell\xi_0')\}E_1(t,x,y;\ell)$ with respect to s. For this we study the properties of roots ξ of $P(\tau,\xi,\eta)=0$. We denote the discriminant of $P(\tau,\xi,\eta)=0$ with respect to ξ by $D(\tau,\eta)$ and write $P(\tau,\xi,\eta)=\prod_{j=1}^m(\tau-\lambda_j(\xi,\eta))$. Then i) $\lambda_i(\xi,\eta)\neq\lambda_j(\xi,\eta)$ if $i\neq j$ and $(\xi,\eta)\neq 0$, ii) $\lambda_i(\xi,\eta)$ is a real-valued homogeneous function of degree 1 and analytic in $R^{n+1}-\{0\}$. Let $P(\sigma_0,\xi_0,\eta_0)=0$, $0\neq(\sigma_0,\xi_0,\eta_0)\in R^{n+2}$. Then there exists λ_k uniquely satisfying $\sigma_0=\lambda_k(\xi_0,\eta_0)$. At first, we study the behavior of roots $\xi=\xi(\tau,\eta;r)$ of $P(\sigma_0+r\tau,\xi,\eta_0+r\eta)=0$ in a neighborhood of $r=0$ satisfying $\xi(\tau,\eta;0)=\xi_0$.

Case I. Assume $D(\sigma_0,\eta_0)\neq 0$. Then $\partial_\xi\lambda_k(\xi_0,\eta_0)\neq 0$ and $\xi(\tau,\eta;r)$ is analytic in a neighborhood of $r=0$. Moreover $R(\tau,\eta)$ is analytic in a neighborhood of (σ_0,η_0). Hence we get

Lemma 1. *If we expand $\xi(\tau,\eta;r)$ as $\xi=\sum_{k=0}^\infty a_k(\tau,\eta)r^k$, then*

$$a_1(\tau,\eta)=\partial_t\xi(\sigma_0,\eta_0)\tau+\sum_{i=1}^n\partial_{\eta_i}\xi(\sigma_0,\eta_0)\eta_i$$
$$=(\partial_\xi\lambda_k(\xi_0,\eta_0))^{-1}(\tau-\textstyle\sum_{j=1}^n\partial_{\eta_j}\lambda_k(\xi_0,\eta_0)\eta_j).$$

Lemma 2. *We expand $R(\sigma_0+r\tau,\eta_0+r\eta)$ as $R=r^{\rho_0}\sum_{k=0}^\infty R_k(\tau,\eta)r^k$, then $R_0(\tau,\eta)$ is a hyperbolic polynomial of degree ρ_0 with respect to τ.*

Case II. Assume $D(\sigma_0,\eta_0)=0$. Then we can represent $P(\tau,\xi,\eta)$ as follows: $P(\tau,\xi,\eta)=\{(\xi-\xi_0)^{m_1}+b_1(\tau,\eta)(\xi-\xi_0)^{m_1-1}+\cdots+b_{m_1}(\tau,\eta)\}P_1(\tau,\eta;\xi)\equiv P_0(\tau,\eta;\xi)P_1(\tau,\eta;\xi)$ where i) $b_i(\sigma_0,\eta_0)=0$ and $b_i(\tau,\eta)$ is holomorphic in a neighborhood of (σ_0,η_0), ii) $P_1(\sigma_0,\eta_0;\xi_0)\neq 0$. We remark that $P_1(\sigma_0,\eta_0;\xi)=0$ may have real multiple roots. Hence the number of roots ξ of $P(\sigma_0+r\tau,\xi,\eta_0+r\eta)=0$ satisfying $\xi(\tau,\eta;0)=\xi_0$ is m_1. We denote them by $\xi_1(\tau,\eta;r),\xi_2(\tau,\eta;r),\cdots,\xi_{m_1}(\tau,\eta;r)$.

Lemma 3. *We expand* $\xi_i(\tau,\eta;r)$ *as* $\xi_i=\sum_{k=0}^{\infty}c_k(\tau,\eta)r^{k/m_1}$, *then*

i) $c_0=\xi_0$, $c_1(\tau,\eta)=\text{const}\,(\tau-\sum_{j=1}^{n}\partial_{\eta_j}\lambda_k(\xi_0,\eta_0)\eta_j)^{1/m_1}$,

ii) *For any* $c_k(\tau,\eta)$ *there exists an integer* p *such that* $c_1(\tau,\eta)^p\times c_k(\tau,\eta)$ *is polynomial.*

Lemma 4. *We write real multiple roots of* $P(\sigma_0,\xi,\eta_0)=0$ *with respect to* ξ *by* $\xi_1^0,\xi_2^0,\cdots,\xi_q^0$, *and their multiplicities by* $m_1,\cdots,m_q$. *Moreover we put* $\sigma_0=\lambda_{k_i}(\xi_i^0,\eta_0)$. *We expand* $R(\sigma_0+r\tau,\eta_0+r\eta)$ *as* $R=\sum_{k=0}^{\infty}R_k(\tau,\eta)r^{\rho_k}(\rho_0<\rho_1<\rho_2<\cdots)$, *then*

$$R_0(\tau,\eta)=\sum_{\deg Q_\beta+\sum_{i=1}^{q}\beta_i/m_i=\rho_0}Q_\beta(\tau,\eta)\cdot\prod_{i=1}^{q}\left(\tau-\sum_{j=1}^{n}\partial_{\eta_j}\lambda_{k_i}(\xi_i^0,\eta_0)\eta_j\right)^{\beta_i/m_i}$$

where $Q_\beta(\tau,\eta)$ *is polynomial and* $\beta_i\geqq 0$ $(i=1,2,\cdots,q)$. *Moreover*

$$R_0(\tau,\eta)\neq 0 \qquad \text{for } \tau=\sigma-i\gamma\ (\gamma>0),(\sigma,\eta)\in R^{n+1}.$$

Remark. If we assume

(A.IV) *If* $P(\sigma,\xi,\eta)=0$ *has real multiple roots with respect to* ξ *for* $0\neq(\sigma,\eta)\in R^{n+1}$, *the number of real multiple roots is at most one,* then $R_0(\tau,\eta)$ is represented as $R_0=r_0(\tau,\eta)(\tau-\sum_{j=1}^{n}\partial_{\eta_j}\lambda_{k_1}(\xi_1^0,\eta_0)\eta_j)^\alpha$ where $r_0(\tau,\eta)$ is a homogeneous hyperbolic polynomial with respect to τ and α is a rational number. Therefore the assumption (A.IV) makes clear the representation of $R_0(\tau,\eta)$, but it is not necessary for the proof of Theorem 1. By Seidenberg's lemma we get the following lemma.

Lemm 5. $R(\tau,\eta)$ *satisfies the following estimate:*

$$\sup_{0<r<\varepsilon}|r^{-\rho_0}R(\sigma_0+r\tau,\eta_0+r\eta)|^{-1}\leqq K(|\tau|+|\eta|)^\beta$$

where $\varepsilon>0$ *and* β *is an constant independent of* (τ,η) *and* r.

Next we state a lemma concerning the distributions.

Lemma 6. *Let* $a=(a_1,a_2,\cdots,a_n)\in R^n$ *and* $\alpha\neq 1,2,3,\cdots$. *Then*

$$\left(\frac{1}{2\pi}\right)^{n+1}\int_{R^{n+1}}\left(\tau-\sum_{j=1}^{n}a_j\eta_j\right)^\alpha e^{i(t\tau+y\eta)}d\sigma d\eta=\frac{e^{\alpha\pi i/2}}{\Gamma(-\alpha)}t_+^{-1-\alpha}\cdot\delta_{(y+ta)},$$

where $t_+^k=t^k$ *for* $t>0$, *and* $=0$ *for* $t<0$.

Under the above preparations, we try to localize $E_1(t,x,y;\ell)$.

$$\begin{aligned}
&e^{-is(t\sigma_0+x\xi_0+y\eta_0-\ell\xi_0')}E_1(t,x,y;\ell)\\
&=s^{-1-m}\sum_{j,k=1}^{\mu}\left(\frac{1}{2\pi}\right)^{n+3}\int_{R^{n+3}}\frac{R_{jk}(\sigma_0+r\tau,\eta_0+r\eta)(\xi_0+r\xi)^{j-1}}{R(\sigma_0+r\tau,\eta_0+r\eta)P_+(\sigma_0+r\tau,\eta_0+r\eta;\xi_0+r\xi)}\\
&\quad\times\frac{B_k(\sigma_0+r\tau,\eta_0+r\eta,\xi_0'+r\xi')}{P(\sigma_0+r\tau,\xi_0'+r\xi',\eta_0+r\eta)}e^{i(t\tau+x\xi+y\eta-\ell\xi')}d\sigma d\xi d\eta d\xi'\\
&=s^{-m-1}\int_{R^{n+3}}G(\sigma_0+r\tau,\xi_0+r\xi,\eta_0+r\eta,\xi_0'+r\xi')\\
&\qquad\qquad\times e^{i(t\tau+x\xi+y\eta-\ell\xi')}d\sigma d\xi d\eta d\xi',
\end{aligned}$$

where $sr=1$. Using Lemma 1 $\sim$ Lemma 4, we can expand G as follows:

$$G=\sum_{i=0}^{\infty}G_i(\tau,\xi,\eta,\xi';P_0)r^{\rho_i},\quad \rho_0<\rho_1<\rho_2<\cdots,\quad P_0=(\sigma_0,\xi_0,\eta_0,\xi_0'),$$

where i) if $D(\sigma_0,\eta_0)\neq 0$, ρ_i are integers, and ii) if $D(\sigma_0,\eta_0)=0$, ρ_i are rational numbers. We define F_k as

$$F_k(t,x,y,\ell;P_0) \tag{3.1}$$
$$=\left(\frac{1}{2\pi}\right)^{n+3}\int_{R^{n+3}} G_k(\tau,\xi,\eta,\xi';P_0)e^{i(t\tau+x\xi+y\eta-\ell\xi')}d\sigma d\xi d\eta d\xi'$$

and put $e_k=-m-1-\rho_k$. Then, using Lemma 5, we get the following

Theorem 1. 1) *For any* $P_0=(\sigma_0,\xi_0,\eta_0,\xi'_0)\in R^{n+3}$ E_1 *has an following asymptotic expansion*

$$e^{-is(t\sigma_0+x\xi_0+y\eta_0-\ell\xi'_0)}E_1(t,x,y;\ell)\sim\sum_{j=0}^{\infty}F_j(t,x,y,\ell;P_0)s^{e_j} \tag{3.2}$$

which has the following property: For every integer N the error

$$s^{-e_N}\Big(e^{-is(t\sigma_0+x\xi_0+y\eta_0-\ell\xi'_0)}E_1-\sum_{j=0}^{N-1}F_j(t,x,y,\ell;P_0)s^{e_j}\Big) \tag{3.3}$$

tends to F_N *in* $\mathcal{D}'(\Omega\times R^1_+)$ *when* $s\to\infty$.

2) $\text{sing supp}\, E_1\supset\bigcup_{P_0\in R^{n+3}}\bigcup_{j=0}^{\infty}\text{supp}\, F_j(t,x,y,\ell;P_0)$.

Remark 1. For obtaining this theorem, we can replace the assumption (A.I) by the less restrictive assumption (A.I)$'$:
(A.I)$'$ $P=\prod_{i=1}^{k}P_i^{m_i}$ *where* P_i *is strictly hyperbolic.*

Remark 2. In the mixed problems it happens the case that $\text{supp}\, F_0\neq\text{supp}\, F_j$ $(j\geqq 1)$. Therefore we must consider other F_j $(j\geqq 1)$ and by this fact we can explain the presence of lateral waves.

At last we calculate any F_k concretely by using Lemma 1 ~ Lemma 4. This is not difficult.

§ 4. Lateral waves. In this section we study the singularities arising from branch points appearing in $G(\tau,\xi,\eta,\xi';P_0)$. If $P(\sigma_0,\xi_0,\eta_0)\neq 0$ or $P(\sigma_0,\xi'_0,\eta_0)\neq 0$, then all $F_j=0$ in $\mathcal{D}'(\Omega\times R^1_+)=0$. Hence we assume $P(\sigma_0,\xi_0,\eta_0)=P(\sigma_0,\xi'_0,\eta_0)=0$ and put $\sigma_0=\lambda_{k_1}(\xi_0,\eta_0)$ and $\sigma_0=\lambda_{k_2}(\xi'_0,\eta_0)$. When $D(\sigma_0,\eta_0)\neq 0$, there is no branch point in $G(\tau,\xi,\eta,\xi')$ and it is easy to calculate G_j. Now we treat the case $D(\sigma_0,\eta_0)=0$. For simplicity we assume (A.IV). We denote a real multiple root of $P(\sigma_0,\xi,\eta_0)=0$ by ζ_0 and assume that ξ_0 is not multiple root of $P(\sigma_0,\xi,\eta_0)=0$. We put $\sigma_0=\lambda_j(\zeta_0,\eta_0)$, then $\partial_\xi\lambda_j(\zeta_0,\eta_0)=0$. Using Lemma 1 ~ Lemma 4, we expand $G(\sigma_0+r\tau,\xi_0+r\xi,\eta_0+r\eta,\xi'_0+r\xi')$ with respect to r. Then it follows

$$G_0=\frac{\text{const}\sum_{j,k=1}^{\mu}R_{jk}(\sigma_0,\eta_0)\xi_0^{j-1}B_k(\sigma_0,\xi'_0,\eta_0)}{R_0(\tau,\eta)(\tau-\langle\text{grad}_{\xi,\eta}\lambda_{k_1}(\xi_0,\eta_0),(\xi,\eta)\rangle)(\tau-\langle\text{grad}_{\xi,\eta}\lambda_{k_2}(\xi'_0,\eta_0),(\xi',\eta)\rangle)}, \tag{3.4}$$

$R_0(\tau,\eta)=r_0(\tau,\eta)(\tau-\sum_{i=1}^{n}\partial_{\eta_i}\lambda_j(\zeta_0,\eta_0)\eta_i)^{\alpha}$.

If in (3.4) $\alpha=0$ or its numerator $=0$, we consider the next term or the more rear term. Then there exists the case where we can find the term G_k such that

$$G_k=\frac{\text{const}\, Q(\tau,\xi,\eta,\xi')(\tau-\sum_{i=1}^{n}\partial_{\eta_i}\lambda_j(\zeta_0,\eta_0)\eta_i)^{\alpha_0}}{r_0(\tau,\eta)^{\alpha_1}(\tau-\langle\text{grad}_{\xi,\eta}\lambda_{k_1}(\xi_0,\eta_0),(\xi,\eta)\rangle)^{\alpha_2}(\tau-\langle\text{grad}_{\xi,\eta}\lambda_{k_2}(\xi'_0,\eta_0),(\xi',\eta)\rangle)^{\alpha_3}}$$

where $\alpha_0\neq 0,1,2,\cdots,\alpha_i>0$ $(i=1,2,3)$, and Q is polynomial. Hence if $F_k\neq 0$, it must be $\partial_\xi\lambda_{k_2}(\xi'_0,\eta_0)>0$ and $\partial_\xi\lambda_{k_1}(\xi_0,\eta_0)<0$. From Theorem 1 we get

$$\text{sing supp } E_1 \supset \text{supp } F_k. \tag{3.5}$$

We explain the meaning of (3.5). For simplicity we treat the case $r_0(\tau,\eta)=\text{const}$. We consider an incident wave travelling from a point source at $(t,x,y)=(0,\ell,0)$ in the direction $-a_0^{-1}(1,a_1,\cdots,a_n)\in R^{n+1}_{x,y}$ where $a_0=(\partial_\xi\lambda_{k_2}(\xi_0',\eta_0))^{-1}$ and $a_i=\partial_{\eta_i}\lambda_{k_2}(\xi_0',\eta_0)a_0$ $(i=1,2,\cdots,n)$. This wave reaches the boundary when $t=a_0\ell$ and its arrival point is $(x,y)=-(0,a_1\ell,\cdots,a_n\ell)$. For this incident wave an ordinary reflected wave S_1 is determined, i.e., $S_1=\{\partial_{\eta_i}\lambda_{k_1}(\xi_0,\eta_0)(t-a_0\ell)+y_i+a_i\ell=0$ $(i=1,2,\cdots,n)$, $\partial_\xi\lambda_{k_1}(\xi_0,\eta_0)(t-a_0\ell)+x=0$, $t-a_0\ell>0$, $\ell>0\}$. Moreover from Lemma 6 we see that there exists a wave S_2 propagating on the boundary, i.e., $S_2=\{\partial_{\eta_i}\lambda_j(\zeta_0,\eta_0)(t-a_0\ell)+y_i+a_i\ell=0$ $(i=1,2,\cdots,n)$, $x=0$, $t-a_0\ell>0$, $\ell>0\}$, and we get

$$\text{supp } F_k=S_1+S_2$$

where $S_1+S_2=\{(t_1+t_2,x_1+x_2,y_1+y_2);(t_i,x_i,y_i)\in S_i, i=1,2\}$. We call S_1+S_2 as lateral wave or branch wave.

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