# 119. Theory of H-valued Fourier Hyperfunctions

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§ 0. Recently the theory of vector valued hyperfunctions has been developed by Ion, P.D.F. and T. Kawai [1]. It has been done by the method of 'soft analysis' in parallel with Sato's theory of hyperfunctions (see Sato, M. [8]). In this paper, we construct the theory of vector valued Fourier hyperfunctions by the method analogous to Kawai's method of constructing the theory of Fourier hyperfunctions (see Kawai, T. [3], [4]). It is known that this theory of vector valued Fourier hyperfunctions is useful in its applications to some problems in the quantum field theory (see Nagamachi, S. and N. Mugibayashi [7]).

We construct the sheaf ${}^H\mathcal{R}$ of $H$-valued Fourier hyperfunctions over $\boldsymbol{D}^n$ as the $n$-th derived sheaf of ${}^H\tilde{\mathcal{O}}$ with support in $\boldsymbol{D}^n$, where $\boldsymbol{D}^n$ is the radial compactification of $\boldsymbol{R}^n$ (see Kawai, T. [4]) and $H$ is a separable complex Hilbert space and ${}^H\tilde{\mathcal{O}}$ is the sheaf of slowly increasing $H$-valued holomorphic functions over $\boldsymbol{D}^n\times\sqrt{-1}\boldsymbol{R}^n$ (see Ito, Y. and S. Nagamachi [2]).

Next we realize $H$-valued Fourier hyperfunctions with supports in a compact set $K$ in $\boldsymbol{D}^n$ as continuous linear operators from $\underset{\sim}{\mathcal{O}}(K)$ to $H$ (as to $\underset{\sim}{\mathcal{O}}(K)$, see Kawai, T. [4]). Namely, we show that the space $H^n_K(V, {}^H\tilde{\mathcal{O}})$ of $H$-valued Fourier hyperfunctions with supports in $K$ is isomorphic to the space $L_b(\underset{\sim}{\mathcal{O}}(K); H)$ of all continuous linear operators from $\underset{\sim}{\mathcal{O}}(K)$ to $H$ equipped with the topology of bounded convergence. We also show that the space $H^n_K(V, {}^H\tilde{\mathcal{O}})$ is isomorphic to the tensor product $H^n_K(V, \tilde{\mathcal{O}})\hat{\otimes}H$ of the space $H^n_K(V, \tilde{\mathcal{O}})$ of scalar valued Fourier hyperfunctions with supports in $K$ and the Hilbert space $H$. These facts are very interesting in comparison with the fact that the spaces of some kinds of vector valued functions and the spaces of vector valued distributions introduced by L. Schwartz have the same properties.

The sheaf ${}^H\mathcal{R}$ is a flabby sheaf and its restriction to $\boldsymbol{R}^n$ coincides with the sheaf ${}^H\mathcal{B}$ of $H$-valued hyperfunctions over $\boldsymbol{R}^n$, and its global sections are stable under Fourier transformations. Hence any $H$-valued hyperfunction on $\boldsymbol{R}^n$ can be extended to an $H$-valued Fourier

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hyperfunction on $\boldsymbol{D}^n$ and then we can consider its Fourier transformation. We can prove an analogue of the Paley-Wiener theorem for Fourier-Carleman-Leray-Sato transformation.

§ 1. In this section, we shall mention some vanishing theorems and duality theorems of cohomology groups with ${}^H\tilde{\mathcal{O}}$ or ${}^H\underset{\sim}{\mathcal{O}}$ as their coefficient sheaves (as to ${}^H\tilde{\mathcal{O}}$ and ${}^H\underset{\sim}{\mathcal{O}}$, see Ito, Y. and S. Nagamachi [2]). We shall also mention relative cohomology groups with support in a compact set $K$ in $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$. They are generalizations of Oka-Cartan Theorem B and Malgrange's theorem and Serre's theorem and Martineau-Harvey's theorem.

**Theorem 1.1** (Oka-Cartan Theorem B). *For any $\tilde{\mathcal{O}}$-pseudoconvex domain $\Omega$ in $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$, we have $H^s(\Omega, {}^H\tilde{\mathcal{O}})=0$ $(s \geqq 1)$.*

**Theorem 1.2.** *Let $K$ be a compact set in $\boldsymbol{D}^n$. Then $H^p(K, {}^H\underset{\sim}{\mathcal{O}}) = 0$ $(p \geqq 1)$.*

**Theorem 1.3** (Malgrange). *Let $\Omega$ be an open set in $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$. Then $H^n(\Omega, {}^H\tilde{\mathcal{O}})=0$.*

**Theorem 1.4** (Serre). *Let $\Omega$ be an open set in $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$. If $\dim H^p(\Omega, {}^H\tilde{\mathcal{O}}) < \infty$ $(p \geqq 1)$, then $(H^j(\Omega, {}^H\tilde{\mathcal{O}}))' \cong H_c^{n-j}(\Omega, {}^H\underset{\sim}{\mathcal{O}})$ for $j=0,1, \cdots, n$, where $H_c^p(\Omega, {}^H\underset{\sim}{\mathcal{O}})$ denotes the $p$-th cohomology group with compact support.*

**Theorem 1.5** (Martineau-Harvey). *Let $K$ be a compact set in $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$ for which $H^p(K, {}^H\underset{\sim}{\mathcal{O}})=0$ holds $(p \geqq 1)$. Let $V$ be an open neighbourhood of $K$. Then we have $H_K^p(V, {}^H\tilde{\mathcal{O}})=0$ $(p \neq n)$ and $H_K^n(V, {}^H\tilde{\mathcal{O}}) = ({}^H\underset{\sim}{\mathcal{O}}(K))'$, where $H_K^p(V, {}^H\tilde{\mathcal{O}})$ denotes the $p$-th relative cohomology group with support in $K$.*

§ 2. In this section, we shall mention an analogue of Runge's theorem.

**Theorem 2.1.** *Let $K$ be a compact set in $\boldsymbol{D}^n$. Then ${}^H\mathcal{P}_* = {}^H\underset{\sim}{\mathcal{O}}(\boldsymbol{D}^n)$ is dense in ${}^H\underset{\sim}{\mathcal{O}}(K)$.*

§ 3. In this section, we shall mention the pure-codimensionality of $\boldsymbol{D}^n$ with respect to ${}^H\tilde{\mathcal{O}}$. Then we shall define the $H$-valued Fourier hyperfunctions and the sheaf ${}^H\mathcal{R}$ of $H$-valued Fourier hyperfunctions over $\boldsymbol{D}^n$ and study some of their properties.

**Theorem 3.1.** *Let $\Omega$ be an open set in $\boldsymbol{D}^n$. Let $V$ be an open set in $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$ which contains $\Omega$ as its closed subset. Then $H_\Omega^p(V, {}^H\tilde{\mathcal{O}}) = 0$ $(p \neq n)$.*

This theorem shows that $\boldsymbol{D}^n$ is purely $n$-codimensional with respect to the sheaf ${}^H\tilde{\mathcal{O}}$. This is an analogue of Sato's theorem on the pure-codimensionality of $\boldsymbol{R}^n$ with respect to the sheaf $\mathcal{O}$ over $\boldsymbol{C}^n$ of germs of holomorphic functions.

**Definition 3.2.** Let $\Omega$ and $V$ be as in the Theorem 3.1. Then we define ${}^H\mathcal{R}(\Omega)$, the space of $H$-valued Fourier hyperfunctions on $\Omega$, by

$H^n_\Omega(V, {}^H\tilde{\mathcal{O}})$. (By the excision theorem the space ${}^H\mathcal{R}(\Omega)$ is independent of the choice of $V$.)

**Theorem 3.3.** *The presheaf $\{{}^H\mathcal{R}(\Omega)\}$ constitutes a flabby sheaf over $\boldsymbol{D}^n$, whose restriction to $\boldsymbol{R}^n$ coincides with the sheaf of $H$-valued hyperfunctions over $\boldsymbol{R}^n$.*

**Definition 3.4.** We denote by ${}^H\mathcal{R}$ the sheaf $\{{}^H\mathcal{R}(\Omega)\}$ over $\boldsymbol{D}^n$ and call it the sheaf of $H$-valued Fourier hyperfunctions over $\boldsymbol{D}^n$.

**Theorem 3.5.** *Let $K$ be a compact set in $\boldsymbol{D}^n$. Then $H^n_K(V, {}^H\tilde{\mathcal{O}}) \cong ({}^H\underset{\sim}{\mathcal{O}}(K))'$.*

**Theorem 3.6.** *Let $\Omega$ be an open set in $\boldsymbol{D}^n$. Let $V$ be an $\tilde{\mathcal{O}}$-pseudoconvex neighbourhood of $\Omega$ such that $V \cap \boldsymbol{D}^n = \Omega$. Let $V_j$ $(j=0,1,\cdots,n)$ be defined by $V_0 = V$ and $V_j = \{z \in V; \operatorname{Im} z_j \neq 0\}$ $(j=1,2,\cdots,n)$. Put $\mathcal{V} = \{V_j\}_{j=0}^n$ and $\mathcal{V}' = \{V_j\}_{j=1}^n$. Then*

$$H^n_\Omega(V, {}^H\tilde{\mathcal{O}}) \cong H^n(\mathcal{V}, \mathcal{V}', {}^H\tilde{\mathcal{O}}) \cong {}^H\tilde{\mathcal{O}}(\textstyle\bigcap_{j=1}^n V_j) / \sum_{j=1}^n {}^H\tilde{\mathcal{O}}(\bigcap_{i\neq j} V_i).$$

Next we will show that $H$-valued Fourier hyperfunctions with supports in a compact set $K$ in $\boldsymbol{D}^n$ can be realized as continuous linear operators from $\underset{\sim}{\mathcal{O}}(K)$ to $H$.

**Theorem 3.7.** *Let $K$ be a compact set in $\boldsymbol{D}^n$. Then $\underset{\sim}{\mathcal{O}}(K)$ is a nuclear space and we have $H^n_K(V, {}^H\tilde{\mathcal{O}}) \cong L_b(\underset{\sim}{\mathcal{O}}(K); H)$.*

**Theorem 3.8.** *Let $K$ be a compact set in $\boldsymbol{D}^n$. Then $H^n_K(V, \tilde{\mathcal{O}})$ is a nuclear space and we have $H^n_K(V, {}^H\tilde{\mathcal{O}}) \cong H^n_K(V, \tilde{\mathcal{O}}) \hat{\otimes} H$.*

**Corollary 3.9.** *Let $K$ be a compact set in $\boldsymbol{D}^n$. Then ${}^H\underset{\sim}{\mathcal{O}}(K) \cong \underset{\sim}{\mathcal{O}}(K) \hat{\otimes} H$.*

§ 4. In this section, we introduce the notion of the Fourier transformation of the elements of ${}^H\mathcal{R}(\boldsymbol{D}^n)$.

**Proposition 4.1.** *If we define $\mathcal{F}\varphi$ by $\int e^{i\langle x,\xi\rangle}\varphi(x)dx$ for $\varphi \in \mathcal{P}_* = \underset{\sim}{\mathcal{O}}(\boldsymbol{D}^n)$, then $\mathcal{F}$ gives a topological isomorphism from $\mathcal{P}_*$ to $\mathcal{P}_*$.*

**Definition 4.2.** *Let $\mu$ be an element of $L_b(\mathcal{P}_*; H)$, then we define $\mathcal{F}^*\mu$ by the formula $\mathcal{F}^*\mu(\varphi) = \mu(\mathcal{F}\varphi)(\forall\varphi \in \mathcal{P}_*)$.*

**Theorem 4.3.** *Every element $\mu \in L_b(\mathcal{P}_*; H)$ can be decomposed as $\mu = \sum_{j=1}^{2^n} \mu_j$, where $\mu_j \in L_b(\underset{\sim}{\mathcal{O}}(K_j); H)$ and $K_j$ denotes the closure of $j$-th quadrant in $\boldsymbol{D}^n$.*

If we define $V_0 = \boldsymbol{D}^n \times \sqrt{-1}I^n$, $V_j = \boldsymbol{D}^n \times \sqrt{-1}\{y \in I^n; y_j \neq 0\}$ (where $I = \{-1 < y < 1\}$), $\mathcal{V} = \{V_j\}_{j=0}^n$ and $\mathcal{V}' = \{V_j\}_{j=1}^n$, we obtain the isomorphism $H^n_{\boldsymbol{D}^n}(\boldsymbol{D}^n \times \sqrt{-1}I^n, {}^H\tilde{\mathcal{O}}) \cong H^n(\mathcal{V}, \mathcal{V}', {}^H\tilde{\mathcal{O}})$ by Leray's theorem (see Komatsu [6]). Thus we can represent any element $\mu$ of $H^n_{\boldsymbol{D}^n}(\boldsymbol{D}^n \times \sqrt{-1}I^n, {}^H\tilde{\mathcal{O}})$ by some element in ${}^H\tilde{\mathcal{O}}(\bigcap_{j=1}^n V_j)$, which we write by $\{\varphi_1, \cdots, \varphi_{2^n}\} = [\varphi]$.

**Definition 4.4.** Using the decomposition of $\mu$ of the Theorem 4.3, we define $\mathcal{F}_s\mu = \{F_j(\zeta)\}$, which is an element of $H^n(\mathcal{V}, \mathcal{V}', {}^H\tilde{\mathcal{O}})$. Here $F_j(\zeta) = (-1)^{j+1}\mu_j(e^{i\langle z,\zeta\rangle})$ ($\operatorname{Im}\zeta$ belongs to the $j$-th open quadrant). We call this $\mathcal{F}_s$ Fourier-Carleman-Leray-Sato transformation.

Then we have

**Theorem 4.5.** $\mathcal{F}_s=\mathcal{F}^*$.

Then we can prove an analogue of the Paley-Wiener theorem for Fourier-Carleman-Leray-Sato transformation (see Ito, Y. and S. Nagamachi [2]).

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