144. *Notes on the Existence of Certain Slit Mappings*

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(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 13, 1975)

The aim of this article is to give a new type of conformal mappings of plane regions bounded by finitely many analytic Jordan curves. This is achieved by making use of a generalized Riemann-Roch theorem shown in [8]. Also we shall mention about some immediate generalizations.

As is well-known, every plane region is conformally equivalent to a parallel slit region. This theorem was carried over the case of Riemann surfaces with positive finite genus by Kusunoki [3]. Other types of canonical regions can be found in [1], [4]–[6] and in Koebe's classical works (see e.g. [2]). The image region with which we shall deal now is of a different sort from those; it is a finite sheeted covering surface of the extended plane whose boundary consists of slits lying over a fixed straight line.

1. Let R be an arbitrary open Riemann surface of genus $g\,(\leqq +\infty)$ and ∂R its Kerékjártó-Stoïlow ideal boundary. Denote by P a fixed regular partition of ∂R such that $P:\partial R=\alpha\cup\beta\cup\gamma$, where $\phi\subseteqq\alpha\subsetneqq\partial R$. We denote by Q the canonical partition of ∂R (see [1]). Let Λ_0 and Λ_0' be two behavior spaces on R which are dual to each other with respect to $\boldsymbol{R}$ (cf. [7]). Suppose that a $(P)\Lambda_0$-divisor $V_P=V(P,\Lambda_0;\beta,m)$ and a $(Q)\Lambda_0'$-divisor $V_Q=V(Q,\Lambda_0';\gamma,n)$ are given. Consider the ordered pair $\Delta=(V_P,V_Q)$ and set $1/\Delta=\Delta^{-1}=(V_Q,V_P)$. The difference $n-m$ of dimensions is called the index of Δ and is denoted by ind Δ. This definition is different from the preceding one ([8], p.15). Because of this, in the present case we may not distinguish two functions with a constant difference. We set $\mathcal{S}(1/\Delta)=\{f\,|\,$(i) f is a single-valued analytic function on R, (ii) df is a multiple of V_Q, (iii) $\Re e\mathfrak{S}_\beta\, f\tau=0$ for every $\tau\in V_P.\}$ and $\mathcal{D}(\Delta)=\{\omega\,|\,\omega$ is a regular analytic differential on R which is a multiple of V_P and satisfies $\Re e\mathfrak{S}_\gamma\, s\omega=0$ for every $ds\in V_Q.\}$. (As for the definitions of $\Re e\mathfrak{S}_\beta\, f\tau$ etc., see [8].)

Now our Riemann-Roch theorem reads:

Theorem 1 ([8]). *For surfaces of finite genus g,*

$$\dim\mathcal{S}(1/\Delta)-\dim\mathcal{D}(\Delta)=\text{ind}\,\Delta-2g+2.$$

One can find a more general form of the Riemann-Roch theorem in [8].

2. In this section we shall show the following theorem as an application of Theorem 1 above.

Theorem 2. *Every plane region R bounded by k analytic Jordan curves can be mapped conformally onto an at most k sheeted covering surface S of the extended plane such that the projection of each boundary contour of S is a line segment lying over the imaginary axis.*

An outline of the proof. Suppose that $\partial R=\bigcup_{i=1}^{k}\beta_i$, β_i being a a contour. For each β_i $(1\leq i\leq k)$ we can take a doubly connected subregion U_i of R such that U_i is conformally equivalent to an annulus $1<|z_i|<r_i\,(<\infty)$, where β_i corresponds to the circle $|z_i|=1$. We may use z_i as a local parameter near β_i.

Next we delete an arbitrarily fixed interior point p_0 of R and get a new open Riemann surface $R_0=R-\{p_0\}$. Denote by β_0 the pointlike ideal boundary of R_0 which arises from "the puncturing p_0". For later use we take a punctured parametric disk U_0 about p_0: $0<|z_0|<r_0\,(<\infty)$.

Let P be the partition of ∂R_0 into ∂R and β_0, while Q denotes the canonical partition of ∂R_0 (it should be noted that we are now concerned with the new Riemann surface R_0). Denote by Λ_{hm} the real Hilbert space of square integrable complex harmonic semiexact differentials on R_0 whose real parts are harmonic measures. We know that Λ_{hm} is a behavior space on R_0 and furthermore it is self-dual with respect to $\boldsymbol{R}$ ([7]). So we can set $\Lambda_0=\Lambda_0'=\Lambda_{hm}$ in the preceding section. In other words, we may deal with the Kusunoki's class $\mathfrak{K}$ of canonical semiexact differentials (cf. [4], p. 340; see also [3]).

In order to construct generalized divisors, we set

$$\sigma_i = ds_i = \begin{cases} dz_0/z_0^{i+1} & \text{on } U_0, \\ 0 & \text{on } U-U_0, \end{cases} \qquad i=1,2,\cdots,k,$$

and

$$\tau_j = \begin{cases} dz_1/z_1 & \text{on } U_1, \\ -dz_{j+1}/z_{j+1} & \text{on } U_{j+1}, \\ 0 & \text{on } U-U_1\cup U_{j+1}, \end{cases} \qquad j=1,2,\cdots,k-1,$$

where $U=\bigcup_{i=0}^{k}U_i$. Then σ_i and τ_j are clearly regular analytic differentials on U. What is more, σ_i is a non-degenerate $(Q)\Lambda_{hm}$-singularity at β_0. Also it is easily seen that τ_j is a non-degenerate $(P)\Lambda_{hm}$-singularity at $\partial R=\partial R_0-\beta_0$. Indeed, $\int_{\partial R}\tau_j=0$ but $\int_{\beta_{j+1}}\tau_j=-\int_{\beta_1}\tau_j=2\pi i\neq 0$ and hence τ_j can never have Λ_{hm}-behavior. We denote by V_P (resp. V_Q) the real vector space spanned by the equivalence classes of τ_j's (resp. σ_i's) modulo Λ_{hm}-behavior. Then V_P (resp. V_Q) is a $(P)\Lambda_{hm}$- (resp. $(Q)\Lambda_{hm}$-) divisor.

Applying Theorem 1 to $\Delta=(V_P,V_Q)$, we know the existence of a non-constant $f_0\in\mathcal{S}(1/\Delta)$. For we have $\operatorname{ind}\Delta=1$. By definition of $\mathcal{S}(1/\Delta)$, (i) f_0 is regular analytic all over R_0, (ii) its differential df_0 is a multiple of V_Q i.e., f_0 has a polar singularity at β_0 of order at most k and has Λ_{hm}-behavior near ∂R, and (iii) f_0 satisfies $\mathfrak{Re}\mathfrak{S}_{\partial R} f_0 \tau_j = 0$, $j=1, 2, \cdots, k-1$. The last condition, together with a part of (ii), means that $\mathrm{Re}\, f_0$ assumes the *same* constant value on each β_i, $i=1, 2, \cdots, k$. As for the discussion concerning the number of sheets of covering $f_0(R_0)$ over $\hat{C}$, we can make use of the argument principle as in Kusunoki [3] (see esp. pp. 256–257). Subtracting an appropriate (complex) constant from f_0 if necessary, we obtain a non-constant meromorphic function f on R which is requested. (We note that $f \in \tilde{\mathfrak{K}}_0(R)$.)

3. It should be noted that the bound of the number of sheets of S in Theorem 2 can be made small. Actually, we can prove the following refinement of Theorem 2.

Theorem 2′. *A plane region R bounded by k analytic Jordan curves can be mapped conformally onto an at most $[(k+1)/2]$ sheeted covering surface S over the Riemann sphere such that each boundary contour of S is a slit on the imaginary axis. Here $[x]$ stands for the greatest integer which does not exceed x.*

For the proof we only need to replace σ_i for $i > [(k+1)/2]$ by $\tilde{\sigma}_i = \sqrt{-1}\, dz_0/z_0^{j+1}$, $j + [(k+1)/2] = i$. Certainly these are linearly independent differentials over the reals.

Also there is no difficulty to extend the above results to the case of positive genus (cf. the proof of Theorem 2′ and [3]). Namely we have

Theorem 3. *Let R be the interior of a compact bordered Riemann surface $\bar{R}$ of genus g with k boundary components. Then R is conformally equivalent to an at most $g + [(k+1)/2]$ sheeted covering surface S over the Riemann sphere whose boundary consists of k line segments lying over the imaginary axis.*

Needless to say, our theorems imply the existence of meromorphic functions whose real parts are $(I)L_1$-principal functions (cf. [1], [5], [6]). We have thus shown that there is an $(I)L_1$-principal function u on R whose conjugate u^* is also single-valued if the order of the preassigned singularity of u is sufficiently large. This result is easily verified by making an appropriate linear combination of $(I)L_1$-principal functions.

References

[1] Ahlfors, L. V., and Sario, L.: Riemann Surfaces. Princeton Univ. Press, Princeton, 382 pp. (1960).
[2] Koebe, P.: Abhandlungen zur Theorie der konformen Abbildung, I–VI. esp. IV: Acta Math., **41**, 305–344 (1918) and V: Math. Z., **2**, 198–236 (1918).
[3] Kusunoki, Y.: Theory of Abelian integrals and its applications to con-

formal mappings. Mem. Coll. Sci. Univ. Kyoto, Ser. A. Math., **32**, 235–258 (1959).

[4] ——: Theory of Functions. Riemann Surfaces and Conformal Mappings (in Japanese). Asakura, Tokyo, 408 pp. (1973).

[5] Rodin, B., and Sario, L.: Principal Functions. Van Nostrand, Princeton, 347 pp. (1968).

[6] Sario, L., and Oikawa, K.: Capacity Functions. Springer, Berlin-Heidelberg-New York, 361 pp. (1969).

[7] Shiba, M.: On the Riemann-Roch theorem on open Riemann surfaces. J. Math. Kyoto Univ., **11**, 495–525 (1971).

[8] ——: A formulation of the Riemann-Roch theorem in terms of differentials with singularities at the ideal boundary. Ibid., **15**, 1–18 (1975).