171. A Characterization of P-Spaces

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P-spaces were introduced by K. Morita [2] (cf. [3]) for an intrinsic characterization of those normal (resp. paracompact) spaces X whose product space $X\times Y$ is normal (resp. paracompact) for each metric space Y. That was a solution of a problem stated by H. Tamano [5].

In the present paper a characterization of P-spaces is given (Theorem 1). Further it is pointed out that this characterization is related to a topological game (Theorem 4). Finally, to each topological space X a metric space $P(X)$ is associated such that e.g. the paracompactness of $X\times P(X)$ implies the paracompactness of $X\times Y$ for each metric space Y (Theorem 8).

Definition ([3], p. 369). Let $\mathfrak{m}$ be a cardinal number $\geqslant 1$. A topological space X is said to be a $P(\mathfrak{m})$-space if for a set I of cardinality $\mathfrak{m}$ and for any family

$$\{G(i_i, \cdots, i_n): (i_1, \cdots, i_n)\in I^n, n\in N\}$$

of open subsets of X such that $G(i_1, \cdots, i_n)\subset G(i_1, \cdots, i_n, i_{n+1})$ for each $(i_1, \cdots, i_n, i_{n+1})\in I^{n+1}$, $n\in N$, there exists a family

$$\{F(i_1, \cdots, i_n): (i_1, \cdots, i_n)\in I^n, u\in N\}$$

of closed subsets of X satisfying the two conditions below:

(a) $F(i_1, \cdots, i_n)\subset G(i_1, \cdots, i_n)$ for each $(i_1, \cdots, i_n)\in I^n$, $n\in N$, and

(b) $\bigcup_{n=1}^{\infty} F(i_1, \cdots, i_n)=X$ for each $(i_1, i_2, \cdots)\in I^N$ such that $\bigcup_{n=1}^{\infty} G(i_1, \cdots, i_n)=X$.

X is said to be a P-space if X is $P(\mathfrak{m})$-space for each cardinal $\mathfrak{m}\geqslant 1$.

Let $\mathfrak{F}$ (resp. $\mathfrak{G}$) denotes the family of all closed (resp. open) subsets of a topological space X.

Theorem 1. *X is a P-space iff there exists a function*

$$F:\bigcup_{n=1}^{\infty}\mathfrak{G}^n\to\mathfrak{F}$$

such that

1.1. *if $(G_1, \cdots, G_n)\in\mathfrak{G}^n$, $n\in N$, then $F(G_1, \cdots, G_n)\subset\bigcup_{k=1}^{n} G_k$, and*

1.2. *if $(G_1, G_2, \cdots)\in\mathfrak{G}^N$ and $\bigcup_{n=1}^{\infty} G_n=X$, then $\bigcup_{n=1}^{\infty} F(G_1, \cdots, G_n)=X$.*

Proof. ($\Rightarrow$) Let X be a P-space. We set $I=\mathfrak{G}$ and $G(G_1, \cdots, G_n)=\bigcup_{k=1}^{n} G_k$ for each $(G_1, \cdots, G_n)\in\mathfrak{G}^n$, $n\in N$. It is clear that $G(G_1, \cdots, G_n)$

$\subset G(G_1, \cdots, G_n, G_{n+1})$ for each $(G_1, \cdots, G_n, G_{n+1}) \in G^{n+1}$, $n \in N$. Thus there exists a family

$$\{F(G_1, \cdots, G_n): (G_1, \cdots, G_n) \in \mathfrak{G}^n, n \in N\}$$

of closed subsets of X so that the conditions (a) and (b) are satisfied. Since $G(G_1, \cdots, G_n) = \bigcup_{k=1}^{n} G_k$, it follows that the function $F: \bigcup_{n=1}^{\infty} \mathfrak{G}^n \to \mathfrak{F}$ has the properties 1.1 and 1.2.

($\Leftarrow$) Let F be a function from $\bigcup_{n=1}^{\infty} \mathfrak{G}^n$ into $\mathfrak{F}$ such that 1.1 and 1.2 are satisfied. Let I be any nonvoid index set and let

$$\{G(i_1, \cdots, i_n): (i_1, \cdots, i_n) \in I^n, n \in N\}$$

be a family of open sets in X such that $G(i_1, \cdots, i_n) \subset G(i_1, \cdots, i_n, i_{n+1})$ for each $(i_1, \cdots, i_n, i_{n+1}) \in I^{n+1}$, $n \in N$. We set

$$F(i_1, \cdots, i_n) = F(G(i_1), G(i_1, i_2), \cdots, G(i_1, \cdots, i_n))$$

for each $(i_1, \cdots, i_n) \in I^n$, $n \in N$. Then we have $F(i_1, \cdots, i_n) \subset G(i_1, \cdots, i_n)$ for each $(i_1, \cdots, i_n) \in I^n$, $n \in N$. Let $(i_1, i_2, \cdots) \in I^N$ and let $\bigcup_{n=1}^{\infty} G(i_1, \cdots, i_n) = X$. Then $\bigcup_{n=1}^{\infty} F(i_1, \cdots, i_n) = \bigcup_{n=1}^{\infty} F(G(i_1), G(i_1, i_2), \cdots, G(i_1, \cdots, i_n)) = X$. Hence X is a P-space.

Theorem 2 which follows is just another variant of Theorem 1.

Theorem 2. *X is a P-space iff there exists a function F defined on the family of all finite sequences $G_1 \subset G_2 \subset \cdots \subset G_n$ of open sets in X such that $F(G_1, \cdots, G_n) \in \mathfrak{F}$, $F(G_1, \cdots, G_n) \subset G_n$ and if $(G_1, G_2, \cdots) \in \mathfrak{G}^N$, $G_n \subset G_{n+1}$ for each $n \in N$ and $\bigcup_{n=1}^{\infty} G_n = X$, then $\bigcup_{n=1}^{\infty} F(G_1, \cdots, G_n) = X$.*

The function F for some of P-spaces can be easily defined and the verifications of conditions 1.1 and 1.2 are not difficult. Here are some examples.

Example 1. Let X be a countably compact space. Then we set $F(G_1, \cdots, G_n) = X$ if $\bigcup_{k=1}^{n} G_k = X$, and we set $F(G_1, \cdots, G_n) = \emptyset$ if $\bigcup_{k=1}^{n} G_k \neq X$.

Example 2. Let X be a σ-compact space, i.e., let $X = \bigcup_{n=1}^{\infty} C_n$ where $C_1, C_2, \cdots$ are compact subsets of X. We set $F(G_1, \cdots, G_n) = \bigcup \left\{C_k : k \leqslant n,\ C_k \subset \bigcup_{m=1}^{n} G_m\right\}$.

Example 3. Let X be a perfectly normal space. Then for each $G \in \mathfrak{G}$ there exists a sequence $(F_1(G), F_2(G), \cdots) \in \mathfrak{F}^N$ such that $G = \bigcup_{n=1}^{\infty} F_n(G)$. We set $F(G_1, \cdots, G_n) = \bigcup_{k=1}^{n} \bigcup_{m=1}^{n} F_k(G_m)$.

Example 4. Let (X, d) be a metric space. We set $F(G_1, \cdots, G_n) = X$

if $\bigcup_{k=1}^{n} G_k = X$ and we set $F(G_1, \cdots, G_n) = \left\{x \in X : d(x, y) \geqslant 1/n \text{ for each } y \in X - \bigcup_{k=1}^{n} G_k\right\}$ if $\bigcup_{k=1}^{n} G_k \neq X$.

Theorem 3. *Let $\mathfrak{m} \geqslant \aleph_0$. Then X is a $P(\mathfrak{m})$-space iff for each family $\mathfrak{A} \subset \mathfrak{G}$ with card $\mathfrak{A} \leq \mathfrak{m}$ there exists a function $F : \bigcup_{n=1}^{\infty} \mathfrak{A}^n \to \mathfrak{F}$ such that*

3.1. $F(A_1, \cdots, A_n) \subset \bigcup_{k=1}^{n} A_k$ *for each* $(A_1, \cdots, A_n) \in \mathfrak{A}^n$, $n \in N$

and

3.2. $\bigcup_{n=1}^{\infty} F(A_1, \cdots, A_n) = X$ *for each* $(A_1, A_2, \cdots) \in \mathfrak{A}^N$ *with* $\bigcup_{n=1}^{\infty} A_n = X$.

The proof of Theorem 3 is similar to the proof of Theorem 1 and thus it is omitted.

Now we shall describe a game associated with P-spaces. Let X be a topological space. Then $\Gamma(X)$ denotes the following infinite positional game with perfect information. There are two players: the first and the second one. The players choose alternatively consecutive terms of a sequence of subsets of X so that each player knows X and first k elements of that sequence when he is choosing the $(k+1)$-st element.

A sequence $(G_1, F_1, G_2, F_2, \cdots)$ of subsets of X is said to be a play of $\Gamma(X)$ if for each $n \in N$ we have

1° $G_n \in \mathfrak{G}$ and G_n is chosen by the first player, and

2° $F_n \in \mathfrak{F}$, $F_n \subset \bigcup_{k=1}^{n} G_k$ and F_n is choosen by the second player.

A play $(G_1, F_1, G_2, F_2, \cdots)$ is a win of the first player if $\bigcup_{n=1}^{\infty} G_n = X$ and $\bigcup_{n=1}^{\infty} F_n \neq X$. A play $(G_1, F_1, G_2, F_2, \cdots)$ is a win of the second player if $\bigcup_{n=1}^{\infty} G_n \neq X$ or if $\bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} F_n = X$. Clearly, each play is a win of exactly one of the players.

The first player obliges the second one to choose F_n bounded by $\bigcup_{k=1}^{\infty} G_k$. The second player wants to choose sufficiently big sets F_n, because if $\bigcup_{n=1}^{\infty} F_n = X$, then he wins. On the other hand the first player tries to extend the set $\bigcup_{k=1}^{n} G_k$ to obtain a cover of X, because if $\bigcup_{n=1}^{\infty} G_n \neq X$, then he loses the play.

A strategy of the first player is a function $s : \{\emptyset\} \cup \bigcup_{n=1}^{\infty} \mathfrak{F}^n \to \mathfrak{G}$. A strategy of the second player is a function $t : \bigcup_{n=1}^{\infty} \mathfrak{G}^n \to \mathfrak{F}$ such that $t(G_1,$

$\cdots, G_n) \subset \bigcup_{k=1}^{n} G_k$ for each $(G_1, \cdots, G_n) \in \mathfrak{G}^n$, $n \in N$.

For each pair (s, t) of strategies there exists a unique play $(G_1, F_1, G_2, F_2, \cdots)$ of $\Gamma(X)$ defined as follows: $G_1 = s(\emptyset)$, $F_1 = t(G_1)$, $G_2 = s(F_1)$, $F_2 = t(G_1, G_2)$, and so on.

A strategy s (resp. t) is said to be winning if the first player (resp. the second player) using s (resp. t) wins every play of $\Gamma(X)$.

According to Theorem 1 and the definition of strategies we have the following game-theoretical characterization of P-spaces.

Theorem 4. *X is a P-space iff the second player has a winning strategy in $\Gamma(X)$.*

Let $\Gamma(X, \mathfrak{A})$ denote the following modification of $\Gamma(X)$. The moves of the first player are restricted to the choice of sets belonging to a given family $\mathfrak{A} \subset \mathfrak{G}$.

Theorem 5. *Let $\mathfrak{m} \geqslant \aleph_0$. Then X is a $P(\mathfrak{m})$-space iff for each family $\mathfrak{A} \subset \mathfrak{G}$ with card $\mathfrak{A} \leqslant \mathfrak{m}$ the second player has a winning strategy in $\Gamma(X, \mathfrak{A})$.*

Theorem 5 is an easy consequence of Theorem 3.

Let us note that the paper [6] contains a sufficient condition for a paracompact space X to get the paracompactness of the product space $X \times Y$ with any paracompact space Y and the condition is nothing else as the existence of a winning strategy in some topological game on X.

To each space X we assign a metric space $P(X)$ defined as follows.

$$P(X) = \left\{ (G_1, G_2, \cdots) \in \mathfrak{G}^N : \bigcup_{n=1}^{\infty} G_n = X \right\}.$$

The canonical base of $P(X)$ consists of all sets $B(U_1, \cdots, U_n) = \{(G_1, G_2, \cdots) \in P(X) : G_1 = U_1, \cdots, G_n = U_n\}$ where $(U_1, \cdots, U_n) \in \mathfrak{G}^n$, $n \in N$. It is well known that the canonical base is σ-discrete, the sets $B(U_1, \cdots, U_n)$ are open and closed and natural distance in $P(X)$ is defined by setting $d((U_1, U_2, \cdots), (V_1, V_2, \cdots)) = 0$ if $U_n = V_n$ for each $n \in N$ and $d((U_1, U_2, \cdots), (V_1, V_2, \cdots)) = 1/n$ if $U_k \neq V_k$ for some $k \in N$ and $n = \min\{k \in N : U_k \neq V_k\}$.

In the proof of Theorem 6 we shall need the following

Lemma. *Let S be a normal, countably paracompact space. Then for each σ-locally finite open cover $\mathfrak{A}$ of S there exists a locally finite open cover $\mathfrak{H}$ of S such that $\{\overline{H} : H \in \mathfrak{H}\}$ refines $\mathfrak{A}$.*

Proof is obvious from [3, Lemma 1.5].

Theorem 6. *If $X \times P(X)$ is normal and countably paracompact, then X is a P-space.*

Proof. Let X be a space such that the product space $X \times P(X)$ is normal and countably paracompact. We shall point out that X admits a function F described by Theorem 1. For $(G_1, \cdots, G_n) \in \mathfrak{G}^n$, $n \in N$, we set $A(G_1, \cdots, G_n)=\left(\bigcup_{k=1}^{n} G_k\right)\times B(G_1, \cdots, G_n)$. Since

$$\mathfrak{B}=\{B(G_1, \cdots, G_n): (G_1, \cdots, G_n) \in \mathfrak{G}^n, n \in N\}$$

is σ-discrete in $P(X)$, it follows that the family

$$\mathfrak{A}=\{A(G_1, \cdots, G_n): (G_1, \cdots, G_n) \in \mathfrak{G}^n, n \in N\}$$

is σ-discrete in $X\times P(X)$. It is easily seen that $\mathfrak{A}$ is an open cover of $X\times P(X)$. By Lemma there exists a locally finite open cover $\mathfrak{H}$ of $X\times P(X)$ such that $\{\overline{H}: H \in \mathfrak{H}\}$ refines $\mathfrak{A}$. For each $(G_1, \cdots, G_n) \in \mathfrak{G}^n$, $n \in N$ and $H \in \mathfrak{H}$ we set

$$D(G_1, \cdots, G_n, H)= \overline{\bigcup\{G \in \mathfrak{G}: G\times B(G_1, \cdots, G_n)\subset H\}}.$$

Then we have $D(G_1, \cdots, G_n, H)\times B(G_1, \cdots, G_n)\subset \overline{H}$ and $H\subset \bigcup\{D(G_1, \cdots, G_n, H)\times B(G_1, \cdots, G_n): (G_1, \cdots, G_n) \in \mathfrak{G}^n, n \in N\}$. Hence it follows that

$$\{D(G_1, \cdots, G_n, H)\times B(G_1, \cdots, G_n): (G_1, \cdots, G_n) \in \mathfrak{G}^n, n \in N, H \in \mathfrak{H}\}$$

is a refinement of $\{\overline{H}: H \in \mathfrak{H}\}$ and it covers $X\times P(X)$. Let $n \in N$ and $(G_1, \cdots, G_n) \in \mathfrak{G}^n$. Then the family $\{D(G_1, \cdots, G_n, H): H \in \mathfrak{H}\}$ is locally finite in X, because $D(G_1, \cdots, G_n, H)\times B(G_1, \cdots, G_n)\subset \overline{H}$ and $\{\overline{H}: H \in \mathfrak{H}\}$ is locally finite in $X\times P(X)$. Let $H \in \mathfrak{H}$ and set $E(G_1, \cdots, G_n, H) = \bigcup\left\{D(G_1, \cdots, G_m, H): m\leq n,\ B(G_1, \cdots, G_m)=B(G_1, \cdots, G_n) \text{ and } D(G_1, \cdots, G_n, H)\subset \bigcup_{k=1}^{n} G_k\right\}$. It is easy to verify that the family $\{E(G_1, \cdots, G_n, H): H \in \mathfrak{H}\}$ is locally finite in X. For each $n \in N$ and $(G_1, \cdots, G_n) \in \mathfrak{G}^n$ we set $F(G_1, \cdots, G_n)= \bigcup\{E(G_1, \cdots, G_n, H): H \in \mathfrak{H}\}$. The set $F(G_1, \cdots, G_n)$ is closed in X, because it is the union of a locally finite family of closed sets. It follows from the definition of $F(G_1, \cdots, G_n)$ that $F(G_1, \cdots, G_n)\subset \bigcup_{k=1}^{n} G_k$. Thus it remains to prove the condition 1.2 of Theorem 1. Let $(G_1, G_2, \cdots) \in P(X)$. We claim that $\bigcup_{n=1}^{\infty} F(G_1, \cdots, G_n) =X$. Let $x \in X$. Then there exists $H \in \mathfrak{H}$ such that $(x, (G_1, G_2, \cdots)) \in H$. Since $\mathfrak{H}$ refines $\mathfrak{A}$, there exists $A(U_1, \cdots U_n) \in \mathfrak{A}$ such that $H \subset A(U_1, \cdots, U_n)$. Since $A(U_1, \cdots, U_n)=\left(\bigcup_{k=1}^{n} U_k\right)\times B(U_1, \cdots, U_n)$ and $(G_1, G_2, \cdots) \in B(U_1, \cdots, U_n)$, we have $U_1=G_1, \cdots, U_n=G_n$. Hence $H \subset A(G_1, \cdots, G_n)$. Since

$$H\subset \bigcup\{D(V_1, \cdots, V_m, H)\times B(V_1, \cdots, V_m): (V_1, \cdots, V_m) \in \mathfrak{G}^m, m \in N\},$$

it follows that there exists $m \in N$ and $(V_1, \cdots, V_m) \in \mathfrak{G}^m$ such that $(x, (G_1, G_2, \cdots)) \in D(V_1, \cdots, V_m, H)\times B(V_1, \cdots, V_m)$. Now again $(G_1, G_2, \cdots) \in B(V_1, \cdots V_m)$ implies $V_1=G_1, \cdots, V_m=G_m$ and therefore $x \in D(G_1, \cdots, G_m, H)$. Moreover $B(G_1, \cdots, G_m)\subset B(G_1, \cdots, G_n)$ and $D(G_1, \cdots, G_m, H)\subset \bigcup_{k=1}^{n} G_k$. We distinguish two cases. *Case* 1: $m\leq n$. Then $B(G_1, \cdots, G_m)=B(G_1, \cdots, G_n)$ and hence $x \in D(G_1, \cdots, G_m, H)\subset E(G_1,$

$\cdots, G_n, H)\subset F(G_1, \cdots, G_n)$. *Case* 2: $m>n$. Then $\bigcup_{k=1}^{n} G_k \subset \bigcup_{k=1}^{m} G_k$ and hence $x \in D(G_1, \cdots, G_m, H)\subset E(G_1, \cdots, G_m, H)\subset F(G_1, \cdots, G_m)$. Therefore $\bigcup_{n=1}^{\infty} F(G_1, \cdots, G_n)=X$.

Let us note that the proof of Theorem 6 is an adaptation of a construction used by K. Morita [3], Lemma 4.5.

Theorem 7. *X is a normal P-space iff the product space $X\times P(X) \times C$ is normal, where C denotes the Cantor Discontinuum.*

Theorem 8. *X is a paracompact P-space iff the space $X\times P(X)$ is paracompact.*

The implications ($\Rightarrow$) of both preceding theorems were proved by K. Morita [3]. Since the normality of $X\times P(X)\times C$ implies the normality and the countable paracompactness of $X\times P(X)$ (cf. [4], Theorem 1.3), the implications ($\Leftarrow$) of the theorems follow from Theorem 6.

References

[1] R. Engelking: Outline of General Topology. Amsterdam (1968).
[2] K. Morita: On the product of a normal space with a metric spaces. Proc. Japan Acad., **39**, 148–150 (1963).
[3] ——: Products of normal spaces with metric spaces. Math. Annalen, **154**, 365–382 (1964).
[4] ——: Note on paracompactness. Proc. Japan Acad., **37**, 1–3 (1961).
[5] H. Tamano: On compactifications. J. Math. Kyoto Univ., **1**, 162–193 (1962).
[6] R. Telgársky: Spaces defined by topological games. Fund. Math., **88**, 15–45 (1975).