32. On Almost-primes in Arithmetic Progressions. III

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(Comm. by Kunihiko KODAIRA, M. J. A., March 12, 1976)

1. In the former papers of this series we have studied the problem of the estimation of the least almost-prime in an arithmetic progression, and we have shown that it is possible to improve drastically Richert's results in this field if we confine ourselves in asymptotic estimations, i.e. for almost all reduced classes modulo a fixed integer or for almost all modulus with a fixed residue.

Now in this note we show a possibility of uniform improvements on Richert's results [1]. Our argument depends on a sieve idea of Selberg [3], and it also depends on a natural assumption concerning the value distribution of the divisor function in arithmetic progressions. By the way we shall give a result on the Twin-Prime Problem, which may throw light on the possibility of the improvement on Selberg's result [3].

In what follows $\tau(n)$, $\mu(n)$, $\varphi(n)$ stand for the divisor, the Moebius and the Euler functions, respectively. Also ε is an arbitrary small fixed positive constant.

2. Let denote by $\mathrm{D}(x;q,l)$ the sum

$$\sum_{\substack{n\leq x\\ n\equiv l\,(\mathrm{mod}\ q)}} \tau(n),$$

and by $\mathrm{D}(x;q)$ the sum

$$\sum_{\substack{n\leq x\\ (n,q)=1}} \tau(n).$$

Then we introduce the following assumptions:

$$\mathcal{D}_\alpha:\quad \mathrm{D}(x;q,l)=\frac{1}{\varphi(q)}\mathrm{D}(x;q)\{1+O((\log x)^{-E})\},$$

uniformly for $q\leq x^\alpha$ and $(q,l)=1$,

$$\mathcal{D}^*_\alpha:\ \sum_{q\leq x^\alpha}\max_{y\leq x}\max_{(q,l)=1}\left|\mathrm{D}(y;q,l)-\frac{1}{\varphi(q)}\mathrm{D}(y;q)\right|\ll x(\log x)^{-K},$$

where E and K are arbitrary but fixed large constant. Then we can show

Theorem 1. *Let*

$$\mathrm{I}(N;q,l;z)=\sum_{\substack{N<n\leq 2N\\ n\equiv l\,(\mathrm{mod}\ q)}}\mu^2(n)\tau(n)\Bigg(\sum_{\substack{d|n\\ d\leq z}}\lambda_d\Bigg)^2,$$

$$I_0(N;q,l;z)=\sum_{\substack{N<n\leqq 2N\\ n\equiv l \pmod q}}\mu^2(n)\left(\sum_{\substack{d\mid n\\ d\leqq z}}\lambda_d\right)^2.$$

Then by an appropriate choice of λ_d and by the assumption $\mathscr{D}_\alpha$, we have

$$\mathrm{I}(N;q,l;z)/\mathrm{I}_0(N;q,l;z)=\left(1+\frac{3\log N}{2\log z}\right)\left(1+O\left(\frac{\log\log N}{\log N}\right)\right),$$

if

$$q\leqq \mathrm{Min}\left\{\left(\frac{N^{1-\varepsilon}}{z^2}\right)^\alpha, N^{(1/2)-\varepsilon}\right\}.$$

Corollary to Theorem 1. *If we assume $\mathscr{D}_\alpha$, then there exists a P_2 (an integer with 2 prime factors at most) such that*

$$\mathrm{P}_2\equiv l \pmod q,\qquad \mu(\mathrm{P}_2)\neq 0,\quad \mathrm{P}_2\leqq \mathrm{Max}\,(q^{(7/3\alpha)+\varepsilon}, q^{2+\varepsilon}),$$

uniformly for all $l, q, (q,l)=1$.

Theorem 2. *Let*

$$\mathrm{I}(N;z)=\sum_{N<n\leqq 2N}\mu^2(n)\mu^2(n+2)(\tau(n)+\tau(n+2))\left(\sum_{\substack{d\mid n(n+2)\\ d\leqq z}}\lambda_d\right)^2,$$

$$\mathrm{I}_0(N;z)=\sum_{N<n\leqq 2N}\mu^2(n)\mu^2(n+2)\left(\sum_{\substack{d\mid n(n+2)\\ d\leqq z}}\lambda_d\right)^2.$$

Then, by an appropriate choice of λ_d and z and by the assumption $\mathscr{D}_\alpha^*$, we have

$$\mathrm{I}(N;z)/\mathrm{I}_0(N;z)\leqq 2(1+4/\alpha)(1+\varepsilon).$$

Remark to Theorem 2. Hooley [2] confirmed $\mathscr{D}_{2/3}$, which, with Theorem 2, gives Selberg's result [3], [4].

3. The actual values of λ_d's may have much interests. In Theorem 1 we have set

$$\lambda_d=\mathrm{Y}_q(z)^{-1}\mu(d)\prod_{p\mid d}(1+2/p)\sum_{\substack{u\leqq z/d\\ (dq,n)=1}}\tau(n)\mu^2(n)/n,$$

if $d\leqq z, (d,q)=1$, and $\lambda_d=0$ otherwise, where

$$\mathrm{Y}_q(z)=\sum_{n\leqq z}\tau(n)\mu^2(n)/n.$$

As for Theorem 2 we have set $z=N^{\alpha/2-\varepsilon}$ and

$$\lambda_d=\mathrm{Y}(z)^{-1}\mu(d)\prod_{p\mid d}\left(\frac{1+1/p-3/p^2}{1-2/p}\right)\sum_{\substack{n\leqq z/d\\ (n,2d)=1}}n^{-1}\tau_3(n)\mu^2(n)\prod_{p\mid n}\left(\frac{1-1/p}{1-2/p}\right),$$

if $d\leqq z, 2\nmid d$, and $\lambda_d=0$ otherwise, where

$$\mathrm{Y}(z)=\sum_{\substack{n\leqq z\\ 2\nmid n}}n^{-1}\tau_3(n)\mu^2(n)\prod_{p\mid n}\left(\frac{1-1/p}{1-2/p}\right),$$

$\tau_3(n)$ being the number of representations of n as a product of 3 positive integers.

References

[1] H. Halberstam and H. E. Richert: Sieve Methods. Academic Press (1974).
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[4] ——: Sieve methods. Proc. Symp. Pure Math., **20**, 311–351 (1971).