95. Solution of R. Telgársky's Problem*)

By Yukinobu YAJIMA
Department of Mathematics, University of Tsukuba

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1. **Introduction.** In [4], R. Telgársky showed that a paracompact space X has a closure-preserving cover by compact sets if X has two order locally finite covers $\{U_\alpha : \alpha \in A\}$ and $\{C_\alpha : \alpha \in A\}$ such that C_α is compact and U_α is an open neighborhood of C_α for each $\alpha \in A$. Order locally finite covers were introduced by Y. Katuta [2]. In the same paper [4], R. Telgársky showed that a paracompact space with two order locally finite covers which are described above is totally paracompact and that a paracompact space with a closure-preserving cover by finite sets is totally paracompact. In these connections, he raised the question of whether or not a paracompact space with a closure-preserving cover by compact sets is totally paracompact ([4] Problem 2). In the present paper, we shall give an affirmative answer to this problem.

All spaces are assumed to be Hausdorff spaces. N denotes the set of all natural numbers.

A space X is said to be *totally paracompact* [1] if each open basis of X contains a locally finite cover of X. A family $\mathfrak{F}$ of subsets of X is said to be *σ-closure-preserving* if $\mathfrak{F}$ is the countable union of families $\{\mathfrak{F}_n\}_{n=1}^{\infty}$ such that $\mathfrak{F}_n$ is closure-preserving for each $n \in N$.

Theorem 1. *If X is a paracompact space with a σ-closure-preserving cover by compact sets, then X is totally paracompact.*

Corollary 2. *If X is a paracompact space with a closure-preserving cover by compact sets, then X is totally paracompact.*

Corollary 2 is an immediate consequence of Theorem 1.

2. **Proof of Theorem 1.** When $\mathfrak{U}$ is a family of subsets of a space X, let $\mathfrak{U}^\# = \bigcup\{U : U \in \mathfrak{U}\}$. Let $\mathfrak{F}$ be a closure-preserving family consisting of compact sets of a space X. For each $x \in X$, $K(x)$ is defined to be $X - \bigcup\{F \in \mathfrak{F} : x \notin F\}$.

When A is a closed subset of X, let

$M_{\mathfrak{F}}(A) = \{x : x \in \mathfrak{F}^\# \cap A,\ K(x)$ is not properly contained in any $K(x')$ for $x' \in A\}$.

We need two lemmas to prove Theorem 1.

Lemma 3 (Potoczny [3]). *Let A be a closed subset of a space X*

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and let $\mathfrak{F}$ be a closure-preserving family by compact sets. Then $M_{\mathfrak{F}}(A)$ can be decomposed into a discrete family by compact sets.

Lemma 4 (Potoczny [3]). *Let $\mathfrak{F}$ be a closure-preserving family by compact sets of a space X, and let $\{V(n): n \in N\}$ be a sequence of open sets of X such that $M_{\mathfrak{F}}(\mathfrak{F}^{\#}) \subset V(1)$ and $M_{\mathfrak{F}}(\mathfrak{F}^{\#} - \{V(i): i=1, \cdots, n\}) \subset V(n+1)$. Then $\mathfrak{F}^{\#} \subset \cup \{V(n): n \in N\}$.*

Proof of Theorem 1. Let $\mathfrak{F} = \bigcup_{n=1}^{\infty} \mathfrak{F}_n$ be a cover of X by compact sets such that $\mathfrak{F}_n$ is closure-preserving for each $n \in N$. Let $\mathfrak{B}$ be an open basis of X. We first construct a sequence of subfamilies $\{\mathfrak{B}_n\}_{n=0}^{\infty}$ of $\mathfrak{B}$ and a sequence of open sets $\{U_n\}_{n=0}^{\infty}$ satisfying the following conditions:

(1) Each $\mathfrak{B}_n$ is locally finite in X.

(2) $M_{\mathfrak{F}_k}(\mathfrak{F}_k^{\#} - \bigcup_{i=1}^{n-1} \mathfrak{B}_i^{\#}) \subset \mathfrak{B}_n^{\#}$ for each $k=1, \cdots, n$.

(3) $L_n \subset U_n \subset \mathrm{Cl}\, U_n \subset \bigcup_{i=1}^{n-1} \mathfrak{B}_i^{\#}$,
where $L_n = \cup \{F \in \bigcup_{i=1}^{n} \mathfrak{F}_i : F \subset \bigcup_{i=1}^{n-1} \mathfrak{B}_i^{\#}\}$.

(4) $\mathrm{Cl}\, U_n \cap \mathfrak{B}_n^{\#} = \emptyset$.

(5) $\mathrm{Cl}\, U_n \subset U_{n+1}$.

Let $\mathfrak{B}_0 = \{\emptyset\}$ and $U_0 = \emptyset$. Assume that $\mathfrak{B}_0, \cdots, \mathfrak{B}_{n-1}$ and $U_0, \cdots, U_{n-1}$ have already been constructed. Since $\bigcup_{i=1}^{n} \mathfrak{F}_i$ is closure-preserving, L_n is closed in X. Since $\mathrm{Cl}\, U_{n-1} \subset \bigcup_{i=1}^{n-1} \mathfrak{B}_i^{\#}$ and X is normal, we can choose an open sets U_n of X such that $\mathrm{Cl}\, U_{n-1} \cup L_n \subset U_n \subset \mathrm{Cl}\, U_n \subset \bigcup_{i=1}^{n-1} \mathfrak{B}_i^{\#}$. By Lemma 3, $M_{\mathfrak{F}_k}(\mathfrak{F}_k^{\#} - \bigcup_{i=1}^{n-1} \mathfrak{B}_i^{\#})$ can be decomposed into a discrete family $\{C_\lambda : \lambda \in \Lambda_k\}$ by compact sets for $k=1, \cdots, n$. Since X is collectionwise normal and each C_λ is compact, we can choose open families

$$\mathfrak{B}_{n,k} = \{B_\lambda^k(j): j=1, \cdots, n(\lambda), \lambda \in \Lambda_k\} \qquad \text{for } k=1, \cdots, n$$

satisfying the following conditions:

(6) $C_\lambda \subset \bigcup_{j=1}^{n(\lambda)} B_\lambda^k(j)$ for each $\lambda \in \Lambda_k$.

(7) $\{\bigcup_{j=1}^{n(\lambda)} B_\lambda^k(j): \lambda \in \Lambda_k\}$ is a discrete family.

(8) Each $\mathfrak{B}_{n,k}$ is a subfamily of $\mathfrak{B}$.

(9) $\mathfrak{B}_{n,k}^{\#} \cap \mathrm{Cl}\, U_n = \emptyset$.

Put $\mathfrak{B}_n = \bigcup_{k=1}^{n} \mathfrak{B}_{n,k}$. It is easy to prove that $\mathfrak{B}_n$ and U_n satisfy the conditions (1)–(5). Put $\tilde{\mathfrak{B}} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$. By (2) and Lemma 4, $\mathfrak{F}_k^{\#} \subset \tilde{\mathfrak{B}}^{\#}$ for each $k \in N$. Hence $\tilde{\mathfrak{B}}$ is an open cover of X. Let $x \in X$. There are $i_0 \in N$ and $F_0 \in \mathfrak{F}_{i_0}$ such that $x \in F_0$. Since F_0 is compact, we can choose $n_0 \in N$ such that $n_0 \geqq i_0$ and $F_0 \subset \bigcup_{i=1}^{n_0-1} \mathfrak{B}_i^{\#}$. Then we have $x \in F_0 \subset L_{n_0} \subset U_{n_0}$. By (4) and (5), $\mathfrak{B}_n^{\#} \cap U_{n_0} = \emptyset$ for each $n \geqq n_0$. Since $\bigcup_{i=1}^{n_0-1} \mathfrak{B}_i$ is locally finite in X, $\tilde{\mathfrak{B}}$ is locally finite at x. Therefore $\tilde{\mathfrak{B}}$ is a locally finite cover of X such that $\tilde{\mathfrak{B}} \subset \mathfrak{B}$. The proof is complete.

References

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