116. A Note on Quasi Metric Spaces

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1. Introduction and notations.

The purpose of this note is to point out errors in a proof and a theorem of Kim [3], and to give a corrected version of the theorem. By a quasi-metric on a set X we mean a non-negative real valued function p on $X \times X$ such that for $x, y, z \in X$ we have $p(x, y)=0$ if and only if $x=y$ and $p(x, y) \leq p(x, z)+p(z, y)$. The set $B(x, p, \varepsilon)=\{y \in X : p(x, y)<\varepsilon\}$ is the p-ball centre x and radius ε. The topology induced on X by p has the family $\{B(x, p, \varepsilon) : x \in X, \ \varepsilon>0\}$ as a base. If p is a quasi-metric on X, its conjugate quasi-metric q on X is given by $q(x, y)=p(y, x)$ for $x, y \in X$. Bitopological concepts which are not defined are taken from Kelly [2].

2. A theorem and an example.

The following result is hinted at by Stoltenberg [6], and proved explicitly in [4].

Theorem 1. *Any quasi metric space whose conjugate quasi metric topology is compact is metrizable.*

Proof. Let T_1 be the topology induced on the set X by the quasi metric p whose conjugate q induces the compact topology T_2 on X. Let U be T_2 open, and $y \in U$. Since (X, T_1, T_2) is pairwise Hausdorff [2], for each $x \in X-U$ there is a T_2 open set U_x and a T_1 open set V_x such that $x \in U_x$, $y \in V_x$ and $U_x \cap V_x=\phi$. Hence $\{U_x : x \in X-U\}$ is a T_2 open cover of $X-U$ which is T_2 compact, and so there is a finite subcover

$U_{x_1}, \cdots, U_{x_n}$. Let $V=\cap\{V_{x_i} : i=1, \cdots, n\}$

It is now easy to prove that either of the metrics d_1 and d_2, given by

$$d_1(x, y)=\frac{1}{2}\{p(x, y)+q(x, y)\} \quad \text{and}$$

$$d_2(x, y)=\max\{p(x, y),\ q(x, y)\} \qquad \text{for } x, y \in X,$$

induces the topology T_1, so that (X, T_1) is metrizable.

The question now arises as to whether the compactness condition of Theorem 1 can be relaxed.

Example 1. This is a modification of an example due to Balanzat [1]. Let X be the set of positive integers and define the non negative real valued function q on $X \times X$ by

$$q(n,m)=\begin{cases}\dfrac{1}{m} & \text{if } n<m\\ 0 & \text{if } n=m\\ 1 & \text{if } n>m.\end{cases}$$

Then $q(n,m)=0$ iff $n=m$, and the following discussion of cases shows that q satisfies the triangle inequality.

Let $n,m,r\in X$, then (i) if $n<m<r$, $q(n,m)=1/m$ while $q(n,r)+q(r,m)=1/r+1$.

(ii) if $n<r<m$, $q(n,m)=1/m$ while $q(n,r)+q(r,m)=1/r+1/m$.

(iii) if $m<r<n$, $q(n,m)=1$ while $q(n,r)+q(r,m)=1+1$.

(iv) if $m<n<r$, $q(n,m)=1$ while $q(n,r)+q(r,m)=1/r+1$.

(v) if $r<m<n$, $q(n,m)=1$ while $q(n,r)+q(r,m)=1+1/m$.

(vi) if $r<n<m$, $q(n,m)=1/m$ while $q(n,r)+q(r,m)=1+1/m$. Thus q is a quasi metric on X, with conjugate p given by

$$p(n,m)=q(m,n)=\begin{cases}1 & \text{if } n<m\\ 0 & \text{if } n=m\\ \dfrac{1}{n} & \text{if } n>m.\end{cases}$$

Let (X,T_1,T_2) be the bitopological space induced by p and q. Then (X,T_2) is not metrizable because it is not Hausdorff. For let $m,n\in X$, $\varepsilon,\delta>0$ and $U=B(m,q,\varepsilon)$ and $V=B(n,q,\delta)$. There is an $r\in X$ such that $r>\max\left\{m,n,\dfrac{1}{\varepsilon},\dfrac{1}{\delta}\right\}$. Then $q(m,r)=1/r<\varepsilon$ and $q(n,r)=1/r<\delta$, so that $r\in U\cap V$. Hence, there is no pair of disjoint T_2 open sets one containing m and the other containing n. Now (X,T_2) is second countable and T_1 so that compactness is equivalent to the Bolzano-Weierstrass property. Let F be any infinite set in X, $n\in F$, and $\varepsilon>0$. Take $m\in X$ such that $m>\max\left\{n,\dfrac{1}{\varepsilon}\right\}$. Since F is infinite there is a $k\in F$ such that $k>m$, and thus $q(n,k)=\dfrac{1}{k}<\dfrac{1}{m}<\varepsilon$, so that $k\in B(n,q,\varepsilon)$. Hence n is a limit point of F, and (X,T_2) is compact. Thus Theorem 1 implies that (X,T_1) is metrizable. Indeed, $B(n,p,1/n)=\{n\}$ for each $n\in X$, so that (X,T_1) is discrete. Then (X,T_2) is a quasi metric space which is not metrizable even though its conjugate topology (X,T_1) is countable and discrete, and hence has the following properties: all the separation properties, Lindelof, second countable, separable, paracompact, locally compact, σ-compact, metacompact, countably paracompact, and is a K-space. Thus no combination of these properties can replace the compactness of Theorem 1.

3. On a paper by Kim.

Kim [3] claims to give a bitopological proof of a theorem of Sion and Zelmer [5]. The following example shows his mistake.

Example 2. Let $X=[0,1]$ and define the real valued function p on $X\times X$ by

$$p(x,y)=\begin{cases} x-y & x\geq y \\ \frac{1}{2}(y-x) & x\leq y. \end{cases}$$

Then p is a quasi metric on X. Now $B(x,p,\varepsilon)=(x-\varepsilon,\ x+2\varepsilon)$ for suitable $x\in X$ and $\varepsilon>0$. Thus p induces the usual topology T_1 on $[0,1]$. Hence (X,T_1) is a regular, compact quasi-pseudo-metric space, and p has conjugate q given by

$$q(x,y)=\begin{cases} y-x & x\leq y \\ \frac{1}{2}(x-y) & x\geq y. \end{cases}$$

So $B(x,q,\varepsilon)=(x-2\varepsilon,\ x+\varepsilon)$ and q induces the usual topology T_2 on $[0,1]$, so that $T_1\subset T_2$. If $d(x,y)=\max\{p(x,y),q(x,y)\}$ then $d(x,y)=|x-y|\neq q(x,y)$ as Kim claims. What can be said is that d induces the same topology as q. In general, nothing can be said about the metrizability of (X,p).

As a corollary to this proof Kim claims the theorem "Any compact quasi metric space is metrizable." The space (X,T_2) of Example 1 shows that he is mistaken. Theorem 1 is a correct version of this result.

References

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