111. On the Completeness of Modified Wave Operators

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The purpose of the present paper is to give a brief account of a proof of the completeness of modified wave operators for long-range scattering.[1)]

Let $\mathcal{H}=L^2(R^N)$, $N\geq 1$ and put

$$(1)\qquad H_1=-\frac{1}{2}\Delta=-\frac{1}{2}\sum_{j=1}^N \partial^2/\partial x_j^2,\qquad H_2=H_1+V,$$

where $V=V(x)\cdot$ denotes the long-range potential satisfying

$$(2)\qquad |\partial^k V(x)|\leq C_k(1+|x|)^{-k-\beta}\qquad \text{with } 1>\beta>1/2,\ C_k>0$$

for $k=0,1,2,\cdots$. Here ∂^k denotes any k-th order partial differentiation in x. Then, H_1 and H_2 are self-adjoint operators in $\mathcal{H}$. For the pair H_1 and H_2, the existence of the modified wave operators

$$(3)\qquad W_D^{\pm}=\operatorname*{s-lim}_{t\to\pm\infty} e^{itH_2}e^{-itH_1-iX(t)},\qquad X(t)=\mathcal{F}^{-1}\left[\int_0^t V(s\xi)ds\cdot\right]\mathcal{F},$$

was proved by Alsholm-Kato [1] and Buslaev-Matveev [2] (cf. also [7]), where $\mathcal{F}$ denotes the Fourier transform in $\mathcal{H}$. Our problem is to prove the completeness of $W_D^{\pm}$. By definition $W_D^{\pm}$ is complete if $\mathcal{R}(W_D^{\pm})=\mathcal{H}_{2,ac}$, where $\mathcal{H}_{2,ac}$ is the absolutely continuous subspace of H_2 and $\mathcal{R}(T)$ denotes the range of an operator T. To prove this, we shall use the stationary modified wave operators $W_\Gamma^{\pm}$ constructed in [6] and the results of Ikebe [4] (or Saitō [9], [10]). (Here and in the sequel, $\Gamma=[a,b]$, $0<a<b<\infty$, is arbitrarily fixed.) For simplicity, we restrict ourselves to considering only W_D^+ in the following, for W_D^- can be dealt with similarly.

We first summarize those results of [6], [7] and [4] which we need in the sequel.

Theorem 1 (cf. [6] and [7]). *Let W_D^+ be as in* (3) *and let W_Γ^+ be the stationary modified wave operator constructed in* [6]. *Then:*

(i) *$W_\Gamma^+=W_D^+E_{1,ac}(\Gamma)$, where $E_{j,ac}$ is the absolutely continuous part of the spectral measure associated with H_j $(j=1,2)$.*

(ii) *For any $x\in\mathcal{X}_1$, $y\in\mathcal{X}_2$ and Borel subsets Δ_1,Δ_2 of Γ,*

$$(4)\qquad (W_\Gamma^+E_{1,ac}(\Delta_1)x,\ E_{2,ac}(\Delta_2)y)_{\mathcal{H}}=\int_{\Delta_1\cap\Delta_2} e_2(\mu;\tilde{x}^+(\mu),y)d\mu.$$

Here $\mathcal{X}_1=\mathcal{F}^{-1}(C_0^\infty(R^N-\{0\}))$ and $\mathcal{X}_2=L_\delta^2(R^N)\equiv L^2(R^N,(1+|x|)^{2\delta}dx)$, δ

1) Recently, Ikebe also proved the completeness of modified wave operators in a way somewhat different from ours (private communication).

being fixed as $\frac{1}{2}<\delta<\frac{1}{2}+\min\left(2\beta-1,\frac{1}{2}\right)$;

(5) $$\tilde{x}^+(\mu)=\operatorname*{l.i.m.}_{\nu\to+0} G^+(\mu+i\nu)x \quad \text{in } L^2(\Gamma;\mathcal{X}_2) \quad \text{for } x\in\mathcal{X}_1,$$

where

(6) $$\begin{cases} G^+(z)=(H_2-z)S^+(z), \\ S^+(z)=i\int_0^\infty e^{-iX(t)}e^{it(z-H_1)}dt \end{cases}$$

for $\operatorname{Im} z>0$ (*as to the existence of the limit in* (5), *see* [6], *Proposition* 2.3); *and*

(7) $$e_j(\mu;u,v)=\frac{1}{2\pi i}(R_j(\mu+i0)u-R_j(\mu-i0)u,v)_{\mathcal{H}}, \qquad \mu>0,$$

for $u,v\in\mathcal{X}_2$. *Here* $R_j(\mu\pm i0)u=\lim_{\nu\to+0}R_j(\mu\pm i\nu)u$ *exist in* $L^2_{-\delta}(R^N)$ *by Ikebe-Saitō* [5] ($R_j(z)=(H_j-z)^{-1}$, $\operatorname{Im} z\neq 0$, $j=1,2$).

Theorem 2 (due to Ikebe [4]). *Let* $j=1$ *or* 2.

(i) *For every* $\mu>0$ *there exists a bounded operator* $\mathcal{F}_j^+(\mu)$ *from* $L^2_\delta(R^N)$ *into* $\boldsymbol{h}=L^2(S^{N-1})$ *satisfying the following conditions*:

(a) *For any* $u,v\in L^2_\delta(R^N)$,

(8) $$e_j(\mu;u,v)=(\mathcal{F}_j^+(\mu)u,\mathcal{F}_j^+(\mu)v)_{\boldsymbol{h}}.$$

(b) *For any* $u\in L^2_\delta(R^N)$, *there exists a positive sequence* $\{r_k\}$ *such that* $r_k\to\infty$ $(k\to\infty)$ *and*

(9) $$\begin{aligned}&(\varphi,\mathcal{F}_j^+(\mu)u)_{\boldsymbol{h}}\\ &\quad=\pi^{-1/2}(2\mu)^{1/4}\lim_{k\to\infty}(\varphi, r_k^{(N-1)/2}e^{i\theta_j^+(r_k,\cdot)}(R_j(\mu+i0)u)(r_k\cdot))_{\boldsymbol{h}}\end{aligned}$$

for any $\varphi\in\boldsymbol{h}$, *where*

(10) $$\theta_1^+(r,\omega)=-\sqrt{2\mu}r, \qquad \theta_2^+(r,\omega)=-\sqrt{2\mu}r+\frac{1}{\sqrt{2\mu}}\int_0^r V(s\omega)ds$$

for $r>0$ *and* $\omega\in S^{N-1}$.

(ii) *Let us define*

(11) $$(\mathcal{F}_j^+u)(\mu)=\mathcal{F}_j^+(\mu)u \quad \text{for } \mu>0 \quad \text{and} \quad u\in L^2_\delta(R^N).$$

Then $\mathcal{F}_j^+$ *can be extended to a partially isometric operator from* $\mathcal{H}$ *onto* $\hat{\mathcal{H}}=L^2((0,\infty);\boldsymbol{h})$ *with the initial set* $\mathcal{H}_{j,ac}$ *and the extended* $\mathcal{F}_j^+$ *satisfies* $\mathcal{F}_j^+E_{j,ac}(B)=\chi_B\mathcal{F}_j^+$ *for any Borel subset* B *of* $(0,\infty)$, *where* χ_B *denotes the characteristic function for* B. *Furthermore put*

(12) $$\mathcal{F}_{j,B}^{+*}f=\int_B \mathcal{F}_j^+(\mu)^*f(\mu)d\mu \qquad \text{for } f\in\hat{\mathcal{H}},\ B\subset(0,\infty),\ 0\notin\bar{B}.$$

Then, $\mathcal{F}_{j,B}^{+*}$ *is a partially isometric operator from* $\hat{\mathcal{H}}$ *into* $\mathcal{H}$ *with the initial set* $L^2(B;\boldsymbol{h})$ *and the final set* $\mathcal{H}_{j,ac}(B)$ *and satisfies* $\mathcal{F}_{j,B}^{+*}=(\mathcal{F}_j^+E_{j,ac}(B))^*$. *Here* $\mathcal{H}_{j,ac}(B)=E_{j,ac}(B)\mathcal{H}$.

Using $\mathcal{F}_j^+$, we can define the following partially isometric operator in $\mathcal{H}$:

(13) $$W_I^+(\Gamma)=\mathcal{F}_2^{+*}\mathcal{F}_1^+E_{1,ac}(\Gamma).$$

By (ii) of Theorem 2, the initial and final sets of $W_I^+(\Gamma)$ are $\mathcal{H}_{1,ac}(\Gamma)$

and $\mathcal{H}_{2,ac}(\Gamma)$, respectively. Thus if we can prove that $W_\Gamma^+ = W_I^+(\Gamma)$, then by (i) of Theorem 1, $\mathcal{R}(W_D^+ E_{1,ac}(\Gamma)) = \mathcal{H}_{2,ac}(\Gamma)$. From this it can be easily seen that $\mathcal{R}(W_D^+) = \mathcal{H}_{2,ac}$, since $\mathcal{H}_{2,ac} = \mathcal{H}_{2,ac}((0,\infty))$, $\mathcal{H} = \mathcal{H}_{1,ac}((0,\infty))$ and $\Gamma = [a,b]$, $0<a<b<\infty$, is arbitrary.

To prove $W_\Gamma^+ = W_I^+(\Gamma)$, we prepare the following theorem.

Theorem 3. *For any $x \in \mathcal{X}_1$, we have*

$$\mathcal{F}_2^+(\mu)\tilde{x}^+(\mu) = \mathcal{F}_1^+(\mu)x \qquad \textit{for a.e. } \mu>0. \tag{14}$$

A complete proof of this theorem will be published later. Here we must be content with the comment that Theorem 3 can be proved under condition (2) in a way similar to the proof of Theorem 3.2.4 of Hörmander [3] using (5) and (9).

Now let us prove $W_\Gamma^+ = W_I^+(\Gamma)$. Let $x \in \mathcal{X}_1$, $y \in \mathcal{X}_2$ and $\Delta_1, \Delta_2 \subset \Gamma$ be fixed. By (ii) of Theorem 1 and (i) of Theorem 2, we have

$$(W_\Gamma^+ E_{1,ac}(\Delta_1)x,\ E_{2,ac}(\Delta_2)y)_{\mathcal{H}} = \int_{\Delta_1\cap\Delta_2} (\mathcal{F}_2^+(\mu)\tilde{x}^+(\mu),\ \mathcal{F}_2^+(\mu)y)_h d\mu. \tag{15}$$

By Theorem 3 and (ii) of Theorem 2, this becomes equal to

$$\begin{aligned}(16)\qquad &\int_{\Delta_1\cap\Delta_2} (\mathcal{F}_1^+(\mu)x,\ \mathcal{F}_2^+(\mu)y)_h d\mu \\ &= \int_{\Delta_2} (\mathcal{F}_2^+(\mu)^*\chi_{\Delta_1}(\mu)\mathcal{F}_1^+(\mu)x,\ y)_{\mathcal{H}} d\mu \\ &= (E_{2,ac}(\Delta_2)\mathcal{F}_2^{+*}\mathcal{F}_1^+ E_{1,ac}(\Delta_1)x,\ y)_{\mathcal{H}} \\ &= (W_I^+(\Gamma)E_{1,ac}(\Delta_1)x,\ E_{2,ac}(\Delta_2)y)_{\mathcal{H}}.\end{aligned}$$

Thus from the fact that $\{E_{j,ac}(\Delta)x \mid x \in \mathcal{X}_j, \Delta \subset \Gamma\}$ is a fundamental subset of $\mathcal{H}_{j,ac}(\Gamma)$, we have $W_\Gamma^+ = W_I^+(\Gamma)$. Therefore, together with the consideration mentioned just before Theorem 3, we have obtained the following theorem.

Theorem 4. *$W_D^\pm$ are complete.*

Our proof mentioned above uses the completeness of $W_I^+(\Gamma)$ and hence (ii) of Theorem 2. But it is possible to prove Theorem 4 only by using (i) of Theorem 2 and to prove more refined and somewhat stronger results than what we have proved above. These results will be treated in subsequent publications together with a complete proof of Theorem 3 under a condition milder than (2). In conclusion, we add a comment that no short-range perturbation does harm the completeness of $W_D^\pm$ (see e.g. Lavine [8]).

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