134. A Characterization of Cliffordian Semigroups

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(Communicated by Kenjiro SHODA, M. J. A., Nov. 12, 1976)

Let S be a semigroup. An element a of S is said to be completely regular if there exists an element x in S such that $axa=a$ and $ax=xa$. A semigroup consisting entirely of completely regular elements is said to be completely regular or Cliffordian. A. H. Clifford [1] proved that completely regular semigroups are semilattices of completely simple semigroups and conversely. Recently, M. Yamada [8] has investigated the regular extensions of a Cliffordian semigroup.

In this short note a new characterization will be given for completely regular elements as well as for completely regular semigroups. For another characterization of Cliffordian semigroups, see [4]. $B(a)$ denote the principal bi-ideal of S generated by the element a of S. For other notations and terminology we refer to [2].

Theorem 1. *An element a of a semigroup S is completely regular if and only if there exists an idempotent element e in S such that*

$$B(a)=B(e). \tag{1}$$

Proof. First, let a be a completely regular element of a semigroup S. Then there is an element x in S so that $a=axa$ and $ax=xa$. Hence we have

$$B(a)=aSa. \tag{2}$$

Let $e=ax=xa$. Then $e^2=e$ and $B(e)=(ax)S(xa)\subseteq B(a)$. Also we have

$$B(a)=(axa)S(axa)=e(aSa)e\subseteq eSe=B(e), \tag{3}$$

and we conclude that (1) holds true.

Conversely, if we suppose (1) for an element a of S, then

$$B(a)=\{a, a^2\}\cup aSa=B(e)=eSe \tag{4}$$

where $e\in E_S$. (4) implies that there is an element s in S such that

$$a=ese. \tag{5}$$

Hence it follows

$$ea=a=ae. \tag{6}$$

On the other hand, (4) implies

$$e=a,\ e=a^2,\quad \text{or}\quad e=ata,\qquad \text{where } t\in S. \tag{7}$$

If $e=ata$, we obtain that $a=a^2ta=ata^2$. Hence

$$a=a^2t(ata^2)=a^2(tat)a^2, \tag{8}$$

that is, $a\in a^2Sa^2$. This holds in the other two cases, too. This means that a is a completely regular element of S (cf. [7], IV. 1.2).

Theorem 2. *A semigroup S is Cliffordian if and only if every* *principal bi-ideal of S can be generated by an idempotent element of S.*

This follows at once from our Theorem 1.

Corollary 1. *A semigroup S is a band of groups if and only if for every element a of S*

$$B(a)=B(e_a), \quad \text{where } e_a \in E_S \tag{9}$$

and, for every couple a, b of elements in S, we have

$$abS=a^2bS \quad \text{and} \quad Sba=Sba^2.$$

This follows from Theorem 7 in [1] and from Theorem 2.

Corollary 2. *A semigroup S is completely simple if and only if it is simple and every principal bi-ideal of S can be generated by an idempotent element of S.*

Corollary 3. *A semigroup S is a completely simple semigroup without zero if and only if it is bisimple and every principal bi-ideal of S can be generated by an idempotent element of S.*

This follows from our Theorem 2 and from Theorem 1 in [6].

References

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