131. *On a Degenerate Oblique Derivative Problem with Interior Boundary Conditions*

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1. Introduction. In this note we shall give the unique solvability theorem for a degenerate oblique derivative problem with a *complex* parameter, by introducing an extra boundary condition and adding an error term to the original boundary condition. The background is some work of Egorov and Kondrat'ev [4] and Sjöstrand [6]. In the non-degenerate case such theorem was obtained by Agranovič and Višic [2]. As an application of this theorem, we shall state some results on the angular distribution of eigenvalues and the completeness of eigenfunctions of a degenerate oblique derivative problem having an extra boundary condition. In the non-degenerate case such results were obtained by Agmon [1].

Let Ω be a bounded domain in $\boldsymbol{R}^n$ ($n\geqq 3$) with boundary Γ of class C^∞. $\bar{\Omega}=\Omega\cup\Gamma$ is a C^∞-manifold with boundary. Let a, b and c be real valued C^∞-functions on Γ, $\boldsymbol{n}$ the unit exterior normal to Γ and α a real C^∞-vector field on Γ. We shall consider the following oblique derivative problem: For given functions f and ϕ defined in Ω and on Γ respectively, find a function u in Ω such that

$$(*)\qquad \begin{cases}(\lambda+\Delta)u=f & \text{in } \Omega,\\ \mathscr{B}u\equiv a\dfrac{\partial u}{\partial \boldsymbol{n}}+\alpha u+(b+ic)u\,|_\Gamma=\phi & \text{on } \Gamma.\end{cases}$$

Here $\lambda=re^{i\theta}$ with $r\geqq 0$ and $0<\theta<2\pi$ and $\Delta=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\cdots+\partial^2/\partial x_n^2$.

If $a(x)\neq 0$ on Γ, then the problem $(*)$ is *coercive* and the unique solvability theorem was obtained by Agranovič and Višik [2].

If $a(x)$ vanishes at some points of Γ, then the problem $(*)$ is *non-coercive*. Egorov and Kondrat'ev [4] studied the problem $(*)$ under the following assumptions (A) and (B):

(A) The set $\Gamma_0=\{x\in\Gamma\,;\,a(x)=0\}$ is an $(n-2)$-dimensional *regular* submanifold of Γ.

(B) The vector field α is *transversal* to Γ_0.

In the case that $a(x)$ changes signs on Γ, they proved the non-existence and non-regularity theorem for the problem $(*)$ and, by introducing an extra boundary condition and adding an error term to the original boundary condition $\mathscr{B}u=\phi$, they succeeded in getting a problem for which they could obtain the existence and regularity theorem, though the unique solvability theorem is not obtained. On the other hand, in the case that $a(x)$ does not change signs on Γ, i.e., in the case that $a(x)\geqq 0$ on Γ, the unique solvability theorem for the problem (*) was obtained by Taira [7].

2. **Results.** In this note, in addition to the assumptions (A) and (B), we introduce the following assumption (C):

(C) On every connected component Γ_0^i of Γ_0 $(i=1,2,\cdots,N)$, we have

$$a=\alpha(a)=\cdots=\alpha^{k_i-1}(a)=0 \quad \text{and} \quad \alpha^{k_i}(a)\neq 0$$

for some *positive* integer k_i.

We divide the set $\{1,2,\cdots,N\}=I^0\cup I^+\cup I^-$ where

$$\begin{cases} i\in I^0 & \text{if and only if } k_i \text{ is even;} \\ i\in I^+ & \text{if and only if } k_i \text{ is odd and } \alpha^{k_i}(a)>0 \text{ on } \Gamma_0^i; \\ i\in I^- & \text{if and only if } k_i \text{ is odd and } \alpha^{k_i}(a)<0 \text{ on } \Gamma_0^i, \end{cases}$$

and we put

$$\Gamma_0^0=\bigcup_{i\in I^0}\Gamma_0^i;\quad \Gamma_0^+=\bigcup_{i\in I^+}\Gamma_0^i;\quad \Gamma_0^-=\bigcup_{i\in I^-}\Gamma_0^i,$$

hence $\Gamma_0=\Gamma_0^0\cup\Gamma_0^+\cup\Gamma_0^-$. Further we put

$$k^0=\max_{i\in I^0}k_i;\quad k^+=\max_{i\in I^+}k_i;\quad k^-=\max_{i\in I^-}k_i,$$

and

$$\delta^0=1/(k^0+1);\ \delta^+=1/(k^++1);\ \delta^-=1/(k^-+1);\quad \delta=\min(\delta^0,\delta^+,\delta^-).$$

For each $s\in\boldsymbol{R}$, we denote the Sobolev spaces on Ω, Γ, Γ_0^+ and Γ_0^- of order s by $H^s(\Omega)$, $H^s(\Gamma)$, $H^s(\Gamma_0^+)$ and $H^s(\Gamma_0^-)$ and their norms by $\|\ \|_{H^s(\Omega)}$, $|\ |_{H^s(\Gamma)}$, $|\ |_{H^s(\Gamma_0^+)}$ and $|\ |_{H^s(\Gamma_0^-)}$ respectively.

Now we can state the main result:

Theorem. *Let $\lambda=re^{i\theta}$ with $r\geqq 0$ and $0<\theta<2\pi$ and let s be any integer $\geqq 2$. Assume that the conditions* (A), (B) *and* (C) *hold. Then we can find the properly supported continuous linear operators $R^+: H^\sigma(\Gamma)\to H^{\sigma-1/2}(\Gamma_0^+)$ and $R^-: H^\sigma(\Gamma_0^-)\to H^{\sigma-1/2}(\Gamma)$ for all $\sigma\in\boldsymbol{R}$ such that if $|\lambda|=r\geqq r_1(\theta)$ for some constant $r_1(\theta)>0$ depending only on θ and s then for any $(f,\phi,u^+)\in H^{s-2}(\Omega)\oplus H^{s-3/2}(\Gamma)\oplus H^{s-3/2+\delta-\delta^+/2}(\Gamma_0^+)$ the problem*

$$\begin{cases} (\lambda+\Delta)u=f & \text{in } \Omega, \\ \mathcal{B}u+R^-w^-\equiv\left(a\dfrac{\partial u}{\partial \boldsymbol{n}}+\alpha u+(b+ic)u\right)\Big|_\Gamma+R^-w^-=\phi & \text{on } \Gamma, \\ R^+(u|_\Gamma)=u^+ & \text{on } \Gamma_0^+ \end{cases}$$

has a unique solution $(u,w^-)\in H^{s-1+\delta}(\Omega)\oplus H^{s-3/2+\delta-/2}(\Gamma_0^-)$ and that the a priori *estimate*

$$\begin{aligned} &\|u\|^2_{H^{s-1+\delta}(\Omega)}+|\lambda|^{s-1+\delta}\|u\|^2_{L^2(\Omega)}+|w^-|^2_{H^{s-3/2+\delta-/2}(\Gamma_0^-)} \\ &\quad+|\lambda|^{s-3/2+\delta-/2}|w^-|^2_{L^2(\Gamma_0^-)}\leqq C_1(\theta)(\|f\|^2_{H^{s-2}(\Omega)}+|\lambda|^{s-2}\|f\|^2_{L^2(\Omega)} \\ &\quad+|\phi|^2_{H^{s-3/2}(\Gamma)}+\lambda^{s-3/2}|\phi|^2_{L^2(\Gamma)}+|u^+|^2_{H^{s-3/2+\delta-\delta^+/2}(\Gamma_0^+)} \\ &\quad+|\lambda|^{s-3/2+\delta-\delta^+/2}|u^+|^2_{L^2(\Gamma_0^+)}) \end{aligned}$$

holds for some constant $C_1(\theta)>0$ depending only on θ and s.

Remark 1. In the case that $\Gamma_0=\Gamma_0^0$, the condition (C) can be weakened (see [7], the condition (C)).

Corollary. *Assume that the conditions* (A), (B) *and* (C) *hold with* $\Gamma_0^- = \phi$. *Let us introduce the linear unbounded operator* $\mathfrak{A}$ *in the Hilbert space* $L^2(\Omega)$ *as follows*:

a) *The domain of* $\mathfrak{A}$ *is* $\mathcal{D}(\mathfrak{A}) = \{u \in H^{1+\delta}(\Omega);\ \Delta u \in L^2(\Omega),\ \mathcal{B}u \equiv a(\partial u/\partial \boldsymbol{n}) + \alpha u + (b+ic)u|_\Gamma = 0$ *and* $R^+(u|_\Gamma) = 0\}$. $(\delta = \min(\delta^0, \delta^+).)$

b) *For* $u \in \mathcal{D}(\mathfrak{A})$, $\mathfrak{A}u = -\Delta u$.

Then the operator $\mathfrak{A}$ *is closed and has the following properties*:

1) *The spectrum of* $\mathfrak{A}$ *is discrete and the eigenvalues of* $\mathfrak{A}$ *have finite multiplicities.*

2) *For any* $\varepsilon > 0$ *there is a constant* $r_2(\varepsilon) > 0$ *depending only on* ε *such that the resolvent set of* $\mathfrak{A}$ *comprises the set* $\{\lambda = re^{i\theta};\ r \geqq r_2(\varepsilon),\ \varepsilon \leqq \theta \leqq 2\pi - \varepsilon\}$ *and that there the resolvent* $(\lambda I - \mathfrak{A})^{-1}$ *satisfies the estimate*

$$\|(\lambda I - \mathfrak{A})^{-1}\| \leqq \frac{C_2(\varepsilon)}{|\lambda|^{(1+\delta)/2}}$$

for some constant $C_2(\varepsilon) > 0$ *depending only on* ε. *In particular, there are only a finite number of eigenvalues outside any angle*: $|\arg \lambda| < \varepsilon$, $\varepsilon > 0$.

3) *The positive axis is a direction of condensation of eigenvalues.*

4) *The generalized eigenfunctions are complete in* $L^2(\Omega)$; *they are also complete in* $\mathcal{D}(\mathfrak{A})$ *in the* $\| \ \|_{H^{1+\delta}(\Omega)}$*-norm.*

Remark 2. Combining the result 2) with Theorema 1–1 of [3], we obtain that the operator-$\mathfrak{A}$ generates an exponential distribution semi-group $U(t)$ which is holomorphic in any sector: $\{z = t + is;\ z \neq 0,\ |\arg z| < \zeta\}$, $0 < \zeta < \pi/2$. Further, arguing as in the proof of Theorem 3.4 in Chap. 1 of [5], it follows that in this sector the estimate $\|U(z)\| \leqq Me^{\omega t} t^{(\delta-1)/2}$ holds for some positive constants M and ω depending only on ζ (cf. [3], Theorema 2–1). Since $0 < \delta < 1$, the semi-group $U(t)$ is unbounded near $t = 0$. But, by using Theorem 3.3 and Theorem 6.8 in Chap. 1 of [5], we can apply Corollary to a mixed problem for the heat equation and obtain the existence and uniqueness theorem.

3. Idea of Proofs. The proof of Theorem is similar to that of Theorem of [7]. First we reduce the problem (*) to the study of a first order pseudodifferential equation $T(\lambda)\varphi = \psi$ on the boundary Γ by means of the Dirichlet problems. Next, by introducing an extra boundary condition $R^+: \mathcal{D}'(\Gamma) \to \mathcal{D}'(\Gamma_0^+)$ and adding an error term $R^-: \mathcal{D}'(\Gamma_0^-) \to \mathcal{D}'(\Gamma)$ to the equation $T(\lambda)\varphi = \psi$, we get a problem

$$\tau(\lambda)\begin{pmatrix} \varphi \\ w^- \end{pmatrix} \equiv \begin{pmatrix} T(\lambda) & R^- \\ R^+ & 0 \end{pmatrix}\begin{pmatrix} \varphi \\ w^- \end{pmatrix} = \begin{pmatrix} \psi \\ u^+ \end{pmatrix},$$

for which we have the existence and regularity theorem. This is the essential step in the proof and proved exactly as in Theorem 1 of [6] (cf. [6], Remark 4.19). Further, using a method of Agmon and Nirenberg as in [7], we show that for $|\lambda|$ sufficiently large the mapping $\tau(\lambda)$ is one to one and onto. Finally we combine these results to get Theorem. The proof of Corollary is the same as that of Corollary of [7].

The details will be given elsewhere.

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