148. *On a Theorem of Ph. Bénilan Concerning Semigroups Systems*

By Akira MARUYAMA
Department of Mathematics, Gakushuin University

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Let X be a real Banach space. By the *duality map* of X into X^*, the dual space of X, we mean the *multivalued mapping* F of X into X^* defined by $Fx=\{f \in X^* ; \langle x, f\rangle=\|x\|^2=\|f\|^2\}$. The *tangent functional* $\tau(x, y)$ on $X \times X$ is defined by $\tau(x, y)=\lim_{t \downarrow 0} t^{-1}(\|x+ty\|-\|x\|)$ for $x, y \in X$, and it is known that $\tau(x, y)$ satisfies the following conditions: (a) $\tau(x, y) \leqq \|y\|$, (b) $\tau(x, y_1+y_2) \leqq \tau(x, y_1)+\tau(x, y_2)$, (c) $\tau(x, ay)=a \cdot \tau(x, y)$ for $a \geqq 0$, (d) $-\tau(x, -y) \leqq \tau(x, y)$, (e) $\tau(x, ax)=a\|x\|$ for real a, (f) $\|x\| \cdot \tau(x, y) = \sup_{f \in Fx} \langle y, f\rangle$. By a *semigroups system* on a closed set $D \subseteq X$, we mean a family $\{S_y(t) ; t \geqq 0, y \in X\}$ of operators from D into itself satisfying

(1) $S_y(0)=I$ (the identity), $S_y(t+s)=S_y(t)S_y(s)$,

(2) $\lim_{t \downarrow 0} S_y(t)x=x$ for $x \in D$,

(3) $\|S_{y_1}(t)x_1-S_{y_2}(t)x_2\| \leqq \|S_{y_1}(s)x_1-S_{y_2}(s)x_2\|+\int_s^t \tau(S_{y_1}(\sigma)x_1-S_{y_2}(\sigma)x_2, y_1-y_2)d\sigma$ for $t \geqq s \geqq 0$ and $x_1, x_2 \in D$ with $y_1, y_2 \in X$.

A multivalued operator A defined on $D(A) \subseteq X$ with values in X is called *accretive* if $\tau(x_1-x_2, y_1-y_2) \geqq 0$ for $y_i \in Ax_i$ $(i=1,2)$, and an accretive operator A is called *m-accretive* if the range $R(I+A)=\{x+y ; y \in Ax, x \in D(A)\}=X$. In this note, we shall discuss the relation between semigroups systems and a family of m-accretive operators. We firstly prove the following

Theorem I. *If A is an m-accretive operator, then the operator $A-y(D(A) \ni x \rightarrow Ax-y)$ is also m-accretive and there exists a semigroups system $\{S_y(t) ; t \geqq 0, y \in X\}$ on the closure $\overline{D(A)}$ of $D(A)$ such that for each $x \in \overline{D(A)}$ we have $S_y(t)x=\lim_{\lambda \downarrow 0}(I+\lambda(A-y))^{-[t/\lambda]} \cdot x$ uniformly in t on every bounded interval of* $[0, \infty)$.

Proof. The proof of (1) and (2) is given by the Crandall-Liggett theorem and (3) is shown in a slightly different form by Bénilan (Thèse, Orsay (1972)). To give a straightforward proof of (3), we shall prepare the following inequality (suggested by I. Miyadera):

$$(3)' \qquad \|S_y(t)x-x_0\| \leqq \|S_y(s)x-x_0\|+\int_s^t \tau(S_y(\sigma)x-x_0, y-y_0)d\sigma$$

for $x \in D(A)$, $y_0 \in Ax_0$ and $t \geqq s \geqq 0$.

For the proof of (3)′, we observe the m-accretiveness of $A-y$ so that we define the *pseudo-resolvent* $J_{\lambda,y}=(I+\lambda(A-y))^{-1}$ for $\lambda>0$ and make use of the fact that $u_k=J_{\lambda,y}^k x$ satisfies the *difference equation*: $\lambda^{-1}(u_k-u_{k-1})+Au_k\ni y$, $u_0=x$. We obtain, by (a)–(f),

$$\lambda^{-1}(\|J_{\lambda,y}^k x-x_0\|-\|J_{\lambda,y}^{k-1}x-x_0\|)\leqq-\tau(J_{\lambda,y}^k x-x_0,\ -\lambda^{-1}(J_{\lambda,y}^k x-x_0-(J_{\lambda,y}^{k-1}x-x_0))$$
$$=-\tau(J_{\lambda,y}^k x-x_0,\ \lambda^{-1}(J_{\lambda,y}^{k-1}x-J_{\lambda,y}^k x))\leqq\lambda^{-1}\langle(J_{\lambda,y}^k x-J_{\lambda,y}^{k-1}x),\ f/\|f\|\rangle$$

for every $f\in F(J_{\lambda,y}^k x-x_0)$. By the accretiveness of $A-y$, there exists a $g\in F(J_{\lambda,y}^k x-x_0)$ such that $\langle-\lambda^{-1}(J_{\lambda,y}^{k-1}x-J_{\lambda,y}^k x)-(y-y_0),g\rangle\leqq 0$. Hence, by (f), we obtain successively

$$\langle\lambda^{-1}(J_{\lambda,y}^k x-J_{\lambda,y}^{k-1}x),\ g/\|g\|\rangle\leqq\langle-\lambda^{-1}(J_{\lambda,y}^{k-1}x-J_{\lambda,y}^k x)+(y_0-y),\ g/\|g\|\rangle$$
$$+\langle y-y_0,\ g/\|g\|\rangle\leqq\langle y-y_0,\ g/\|g\|\rangle\leqq\tau(J_{\lambda,y}^k x-x_0,\ y-y_0),$$
$$\|J_{\lambda,y}^k x-x_0\|\leqq\|J_{\lambda,y}^{k-1}x-x_0\|+\lambda\tau(J_{\lambda,y}^k x-x_0,\ y-y_0)$$
$$=\|J_{\lambda,y}^{k-1}x-x_0\|+\int_{k\lambda}^{(k+1)\lambda}\tau(J_{\lambda,y}^k x-x_0,\ y-y_0)d\sigma.$$

Let $t\geqq s\geqq 0$ and add the latter for $k=[s/\lambda]+1,\cdots,[t/\lambda]$. Then

$$\|J_{\lambda,y}^{[t/\lambda]}x-x_0\|\leqq\|J_{\lambda,y}^{[s/\lambda]}x-x_0\|+\int_{([s/\lambda]+1)\lambda}^{([t/\lambda]+1)\lambda}\tau(J_{\lambda,y}^{[\sigma/\lambda]}x-x_0,\ y-y_0)d\sigma.$$

Since $\|J_{\lambda,y}^{[t/\lambda]}x-x_0\|\leqq\|J_{\lambda,y}^{[t/\lambda]}x-J_{\lambda,y}^{[t/\lambda]}x_0\|+\|J_{\lambda,y}^{[t/\lambda]}x_0-x_0\|\leqq\|x-x_0\|+t\,\|z\|$ with a $z\in Ax_0-y$, we can get (3)′ from the Lebesque-Fatou lemma and the upper-semicontinuity of the functional τ by taking the $\limsup_{\lambda\downarrow 0}$. Then, taking $x_0=J_{\lambda,y_2}^m x_2$ and $y_0=\lambda^{-1}(J_{\lambda,y_2}^{m-1}x_2-J_{\lambda,y_2}^m x_2)+y_2$ in (3)′, we have

$$\|S_{y_1}(t)x_1-J_{\lambda,y_2}^m x_2\|\leqq\|S_{y_1}(s)x_1-J_{\lambda,y_2}^m x_2\|$$
$$+\int_s^t\tau(S_{y_1}(\sigma)x_1-J_{\lambda,y_2}^m x_2,\ y_1-\lambda^{-1}(J_{\lambda,y_2}^{m-1}x_2-J_{\lambda,y_2}^m x_2)-y_2)d\sigma,$$

where the integrand is, by (b), (a) and (e), smaller than

$$\lambda^{-1}(\|S_{y_1}(t)x_1-J_{\lambda,y_2}^{m-1}x_2\|-\|S_{y_1}(\sigma)x_1-J_{\lambda,y_2}^m x_2\|)+\tau(S_{y_1}(\sigma)x_1-J_{\lambda,y_2}^m x_2,\ y_1-y_2).$$

Let $0\leqq a<b<\infty$ and $k=[a/\lambda]$, $i=[b/\lambda]$. Then, adding the above inequality for $m=k+1,\cdots,i$ and taking the $\limsup_{\lambda\downarrow 0}$, we have

$$\int_a^b(\|S_{y_1}(t)x_1-S_{y_2}(\xi)x_2\|-\|S_{y_1}(s)x_1-S_{y_2}(\xi)x_2\|)d\xi\leqq\int_s^t(\|S_{y_1}(\sigma)x_1-S_{y_2}(a)x_2\|$$
$$-\|S_{y_1}(\sigma)x_1-S_{y_2}(b)x_2\|)d\sigma+\int_s^t d\sigma\int_a^b\tau(S_{y_1}(\sigma)x_1-S_{y_2}(\xi)x_2,\ y_1-y_2)d\xi$$

and so we obtain (3) by applying Lemma 1.2 in Bénilan's Thèse.

Now we are able to give a straightforward proof, based upon the idea of the *product integral*, of the following theorem of Bénilan:

Theorem II. *Given a semigroups system $\{S_y(t);t\geqq 0,y\in X\}$ on a closed set D in X, there exists exactly one m-accretive operator A such that $\overline{D(A)}=D$ and for all $x\in D$ we have $S_y(t)x=\lim_{\lambda\downarrow 0}(I+\lambda(A-y))^{-[t/\lambda]}\cdot x$ uniformly on every bounded interval of* $[0,\infty)$.

Proof. Let $\mathcal{T}_x$ be a mapping from $C([0,T];X)$ into itself given by $\mathcal{T}_x: C([0,T];X)\ni u=u(t)\rightarrow\prod_0^t S_{y-u(\tau)}(d\tau)x$, where the *product integral* $\prod_0^t S_{y-u(\tau)}(d\tau)x$ is defined as below: Let $\{\mathcal{P}_\alpha: 0=t_0^\alpha<t_1^\alpha<\cdots<t_{n(\alpha)}^\alpha$

$=t\,;\,\alpha\in\mathcal{A}\}$ be a net of partitions of $[0,t]$ with α contained in a directed set $\mathcal{A}$ such that $\lim_{\alpha\in\mathcal{A}}\max_{1\leq i\leq n(\alpha)}|t_i^\alpha-t_{i-1}^\alpha|=0$. Then we can show that the net of product $\prod_{i=1}^{n(\alpha)}S_{y-u(\tau_i^\alpha)}(t_i^\alpha-t_{i-1}^\alpha)\cdot x$, associated with the partition $\mathcal{P}_\alpha$, strongly converges to a certain point of X whenever $\lim_{\alpha\in\mathcal{A}}\max_{1\leq i\leq n(\alpha)}|t_i^\alpha-t_{i-1}^\alpha|=0$, irrespective of the choice of points $\tau_i^\alpha\in[t_{i-1}^\alpha,t_i^\alpha)$. In fact, let $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ be partitions of $[0,t)$, and let $\mathcal{P}_\gamma$ be the partition obtained by superposing these two partitions. Then we have, by (1),

$$\prod_{i=1}^{n(\alpha)}S_{y-u(\tau_i^\alpha)}(t_i^\alpha-t_{i-1}^\alpha)x-\prod_{i=1}^{n(\beta)}S_{y-u(\tau_i^\beta)}(t_i^\beta-t_{i-1}^\beta)x$$
$$=\prod_{j=1}^{n(\gamma)}S_{y-u(\sigma_j^{\gamma,\alpha})}(t_j^\gamma-t_{j-1}^\gamma)x-\prod_{j=1}^{n(\gamma)}S_{y-u(\sigma_j^{\gamma,\beta})}(t_j^\gamma-t_{j-1}^\gamma)x=I_{n(\gamma)},$$

where $\sigma_j^{\gamma,\alpha}\in\{\tau_i^\alpha\,;\,1\leqq n\leqq(\alpha)\}$ and $\sigma_j^{\gamma,\beta}\in\{\tau_i^\beta\,;\,1\leqq i\leqq n(\beta)\}$. Hence we have, by (3) and (a),

$$\begin{aligned}I_{n(\gamma)}=I_{n(\gamma)-1}+\int_{t_{n(\gamma)-1}^\gamma}^{t_{n(\gamma)}^\gamma}\tau\Big(&S_{y-u(\sigma_{n(\gamma)-1}^{\gamma,\alpha})}(s-t_{n(\gamma)-1}^\gamma)\prod_{j=1}^{n(\gamma)-1}S_{y-u(\sigma_j^{\gamma,\alpha})}(t_j^\gamma-t_{j-1}^\gamma)x\\ &-S_{y-u(\sigma_{n(\gamma)-1}^{\gamma,\beta})}(s-t_{n(\gamma)-1}^\gamma)\prod_{j=1}^{n(\gamma)-1}S_{y-u(\sigma_{n(\gamma)-1}^{\gamma,\beta})}(t_j^\gamma-t_{j-1}^\gamma)x,\\ &-u(\sigma_{n(\gamma)-1}^{\gamma,\alpha})+u(\sigma_{n(\gamma)-1}^{\gamma,\beta})\Big)ds\\ \leqq I_{n(\gamma)-1}+|t_{n(\gamma)}^\gamma&-t_{n(\gamma)-1}^\gamma|\cdot\|u(\sigma_{n(\gamma)-1}^{\gamma,\alpha})-u(\sigma_{n(\gamma)-1}^{\gamma,\beta})\|\\ \leqq\sum_{j=1}^{n(\gamma)}|t_j^\gamma-t_{j-1}^\gamma&|\cdot\|u(\sigma_{j-1}^{\gamma,\alpha})-u(\sigma_{j-1}^{\gamma,\beta})\|.\end{aligned}\tag{4}$$

Thus $\prod_0^t S_{y-u(\tau)}(d\tau)x=\lim_{\gamma\in\mathcal{A}}\prod_{j=0}^{n(\gamma)}S_{y-u(\tau_j^\gamma)}(t_j^\gamma-t_{j-1}^\gamma)x$ exists.

The above obtained mapping $\mathcal{T}_x$ satisfies the following inequality

$$\begin{aligned}&\|\mathcal{T}_xu-\mathcal{T}_xv\|_{C([0,T];\,x)}\\ &\leqq\int_0^T\|u(t)-v(t)\|\,dt\Big(\|u-v\|_{C([0,T;\,X)}=\sup_{t\in[0,T]}\|u(t)-v(t)\|\Big)\end{aligned}\tag{5}$$

so that, we have

$$\|\mathcal{T}_x^nu-\mathcal{T}_x^nv\|_{C([0,T]:\,X)}\leqq(n\,!)^{-1}T^n\,\|u-v\|_{C([0,T];\,X)}\qquad(n=1,2,\cdots).\tag{5$'$}$$

For the proof of (5), we prove, similarly as (4),

$$\Big\|\prod_{j=1}^{n(\alpha)}S_{y-u(\tau_j^\alpha)}(t_j^\alpha-t_{j-1}^\alpha)x-\prod_{j=1}^{n(\alpha)}S_{y-v(\tau_j^\alpha)}(t_j^\alpha-t_{j-1}^\alpha)x\Big\|$$
$$\leqq\sum_{i=1}^{n(\alpha)}(t_i^\alpha-t_{i-1}^\alpha)\cdot\|u(\tau_i^\alpha)-v(\tau_i^\alpha)\|$$

and take the $\lim_{\alpha\in\mathcal{A}}$.

By virtue of (5)$'$, $\mathcal{T}_x^n$ is a strictly contractive mapping in $C([0,T]\,;X)$ for sufficiently large n and so there exists one and only one fixed point of $\mathcal{T}_x$ in $C([0,T]\,;X)$. Hence, for any $y\in X$, x and $x'\in X$, there exists the unique solutions u_0 and v_0 in $C([0,T]\,;X)$ respectively of the product integral equations of the Volterra type $u_0(t)=\prod_0^t S_{y-u_0(\tau)}(d\tau)x$ and $v_0(t)=\prod_0^t S_{y-v_0(\tau)}(d\tau)x'$. Since T was arbitrary, these solutions are global ones and each of them has the limit as $t\to\infty$, which is the value of the resolvent of an m-accretive operator that we are intending to find. The proof: Let $T_y(t)x$ be the mapping given by $x\to u_0(t)$. Then we can prove that $T_y(t)$ is a semigroup on D and satisfies

$$(6)\qquad \|T_y(t)-T_y(t)x'\|\leqq\|T_y(s)x-T_y(s)x'\|-\int_s^t\|T_y(\sigma)x-T_y(\sigma)x'\|\,d\sigma$$

for $t\geqq s\geqq 0$. To this purpose, we prove, similarly as (5)

$$(7)\qquad \begin{aligned}&\|\mathcal{T}_xu(t)-\mathcal{T}_{x'}v(t)\|\leqq\|\mathcal{T}_xu(s)-\mathcal{T}_{x'}v(s)\|\\ &\qquad+\int_s^t\tau(\mathcal{T}_xu(\sigma)-\mathcal{T}_{x'}v(\sigma),\,-u(\sigma)+v(\sigma))d\sigma,\end{aligned}$$

where u and $v\in C([0,T];X)$ and $t\geqq s\geqq 0$. Hence we have

$$\begin{aligned}&\|\mathcal{T}_x^nu(t)-\mathcal{T}_{x'}^nv(t)\|\leqq\|\mathcal{T}_x^nu(s)-\mathcal{T}_{x'}^nv(s)\|\\ &\qquad+\int_s^t\tau(\mathcal{T}_x^nu(\sigma)-\mathcal{T}_{x'}^nv(\sigma),\,-\mathcal{T}_x^{n-1}u(\sigma)+\mathcal{T}_{x'}^{n-1}v(\sigma))d\sigma,\qquad(n=1,2,\cdots).\end{aligned}$$

Thus we obtain (6) by (e) and by letting $n\to\infty$. In fact, the $\lim_{n\to\infty}\mathcal{T}_x^nu=u_0$ is the unique fixed point $T_y(t)x$ of $\mathcal{T}_x$, i.e., $u_0(t)=\prod_0^t S_{y-u_0(\tau)}(d\tau)x$. The obtained $T_y(t)$ is a contraction operator with Lipschitz constant e^{-t}, since $\|T_y(t)x-T_y(t)x'\|+\int_0^t\|T_y(\sigma)x-T_y(\sigma)x'\|\,d\sigma$ is monotone increasing in t by (6). We next show that $T_y(t)$ has the property (1). In fact, we have

$$\begin{aligned}T_y(t+s)x=u_0(t+s)&=\prod_0^{t+s}S_{y-u_0(\tau)}(d\tau)x=\prod_s^{t+s}S_{y-u_0(\tau)}(d\tau)\cdot\prod_0^{s}S_{y-u_0(\sigma)}(d\sigma)x\\ &=\prod_0^{t}S_{y-u_0(\tau+s)}(d\tau)u_0(s)=\prod_0^{t}S_{y-u_0(\tau+s)}(d\tau)T_y(t)x.\end{aligned}$$

On the other hand, we have $T_y(t)T_y(s)x=\prod_0^t S_{y-\omega_s(\tau)}(d\tau)T_y(s)x=\omega_s(t)$ and hence, by the uniqueness of the solution of the product integral equation, we obtain $\omega_s(t)=u_0(t+s)$, i.e., $T_y(t)T_y(s)x=T_y(t+s)x$. Thus, by the Lipschitz constant e^{-t} of $T_y(t)$, we have $\lim_{t\to\infty}u_0(t)=\lim_{t\to\infty}T_y(t)x=x_0$, where $x_0=T_y(t)x_0$. Hence we must have

$$(8)\qquad x_0=S_{y-x_0}(t)x_0\qquad\text{for all } t\geqq 0,$$

because x_0 is the solution of $u(t)=\prod_0^t S_{y-u(\tau)}(d\tau)x_0$, $u(0)=x_0$. We put

$$(9)\qquad A=\{\{x,y\};\,S_y(t)x=x\text{ for all }t\geqq 0\}.$$

The accretiveness of A is easily seen by putting $S_{y_1}(t)x_1=x_1$ and $S_{y_2}(t)x_2=x_2$ in (3) and so the m-accretiveness of A is easily seen from (8). Hence by (3) we obtain (3)′ whenever $\{x_0,y_0\}\in A$. It is proved by Bénilan that if $S_y(t)x$ satisfies (3)′ then the orbit of $S_y(t)x$ is contained in $\overline{D(A)}$. Therefore, we can say that $\overline{D(A)}=D$. In order to complete our proof, we have to show that $S_y(t)x=\lim_{\lambda\downarrow 0}(I+\lambda(A-y))^{-[t/\lambda]}\cdot x$. However, this proof is easily obtained similarly to that of (3)′.