4. On Discontinuous Groups Acting on a Real Hyperbolic Space. II

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0°. Let $G^{(n)}$ be the motion group of a real n-dimensional hyperbolic space $\boldsymbol{H}$. In 1° we apply the two theorems in the preceding note [1] to give explicit fundamental domains and fundamental relations for arithmetic discrete subgroups of $G^{(n)}$ where $4 \leqq n \leqq 9$. In 2° we show some examples of discrete subgroups by giving fundamental domains in case $n=3$.

1°. We define an arithmetic group Γ of $G^{(n)}$. Let $\boldsymbol{H}$ be the upper half space $\{\xi={}^t(\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n \mid \xi_n \geqq 0\}$ of $\boldsymbol{R}^n$ with metric form $ds^2 = \left(\sum_{j=1}^n d\xi_j^2\right)/\xi_n^2$. Let Q be the matrix of degree $(n+1)$

$$\begin{pmatrix} 1_{n-1} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

where 1_{n-1} means the unit matrix of degree $n-1$. Let X_Q be a connected component of the hypersurface $\{x={}^t(x_1, \cdots, x_{n+1}) \in \boldsymbol{R}^{n+1} \mid {}^tx.Q.x = -1\}$ of $\boldsymbol{R}^{n+1}$. Then the motion group $G^{(n)}$ is the subgroup

$$\{g \in GL(n+1, \boldsymbol{R}) \mid {}^tg.Q.g=Q,\ g(X_Q)=X_Q\}$$

of $GL(n+1, \boldsymbol{R})$. Its action on $\boldsymbol{H}=\boldsymbol{H}^n$ is given by $g.\xi=\eta$ for

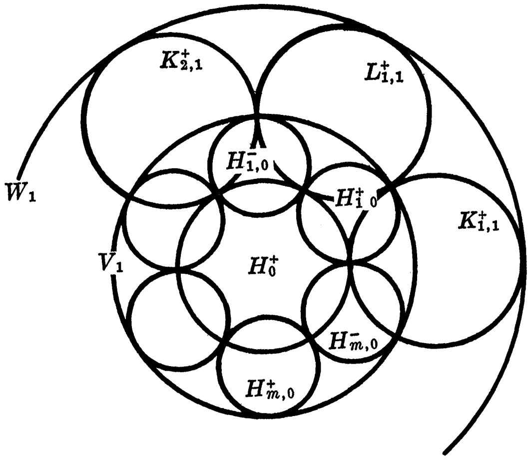
$$g=\begin{bmatrix} \sigma & \gamma_1 & \gamma_2 \\ {}^t\delta_1 & \alpha_1 & \alpha_2 \\ {}^t\delta_2 & \alpha_3 & \alpha_4 \end{bmatrix} \in G^{(n)},$$

$\sigma \in M(n-1, \boldsymbol{R})$, γ_i, $\delta_i \in \boldsymbol{R}^{n-1}$ ($i=1$ or 2), $\alpha_i \in \boldsymbol{R}$ ($1 \leqq i \leqq 4$), ${}^t\xi=({}^t\xi', \xi_n)$, $\xi' \in \boldsymbol{R}^{n-1}$ where η is defined by ${}^t\eta=({}^t\eta', \eta_n)$, $\eta' \in \boldsymbol{R}^{n-1}$, $\eta'=\left({}^t\delta_2\xi'+\frac{1}{2}({}^t\xi\xi)\alpha_3+\alpha_4\right)^{-1}\left(\sigma\xi'+\frac{1}{2}({}^t\xi\xi)\gamma_1+\gamma_2\right)$ and $\eta_n=\left({}^t\delta_2\xi'+\frac{1}{2}({}^t\xi\xi)\alpha_3+\alpha_4\right)^{-1}\xi_n$. We denote by $\Gamma^{(n)}$ the group $G^{(n)} \cap SL(n+1, \boldsymbol{Z})$. From now on we assume that $4 \leqq n \leqq 9$. We construct a fundamental domain F fit for $\Gamma^{(n)}$. We denote by Γ^∞ the subgroup of $\Gamma=\Gamma^{(n)}$ fixing the point at infinity considered to be contained in $\partial\boldsymbol{H}$ and by Δ the set $\{\xi={}^t(\xi_1, \cdots, \xi_n) \in \boldsymbol{H} \mid \xi_1 + \xi_2 < 1, \xi_1 > \xi_3, \xi_2 > \xi_3, \xi_3 > \xi_4 > \cdots > \xi_{n-1} > 0\}$. Then Δ is a fundamental domain for Γ^∞, namely $\bigcup_{g \in \Gamma^\infty} g\bar{\Delta}=\boldsymbol{H}$ and $g\Delta \cap \Delta=\phi$ for any $g \in \Gamma^\infty-\{e\}$ where e means the unit element of $G^{(n)}$. For each $g \in \Gamma-\Gamma^\infty$ we denote by D_g the set $\left\{\xi\in \boldsymbol{H}\,\middle|\,\left|\xi+\frac{1}{\alpha_3}\binom{\delta_2}{0}\right|>\sqrt{\frac{2}{\alpha_3}}\right\}$. Note that $\alpha_3\geqq 1$ for $g\in\Gamma-\Gamma^\infty$ and $\alpha_3=0$ for $g\in\Gamma^\infty$. The boundary of D_g is totally geodesic. Now let F denote the set $\Delta\cap\bigcap_{g\in\Gamma-\Gamma^\infty} D_g$. Then by standard arguments F is a fundamental domain for Γ. The assumption $n\leqq 9$ means that F is equal to $\Delta\cap D_{g_1}$, where $g_1=\begin{bmatrix}\rho & 0 & 0\\ 0 & 1_{n-3} & 0\\ 0 & 0 & \rho\end{bmatrix}$, ρ being $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, so that $D_{g_1}=\{\xi\in\boldsymbol{H}\,|\,|\xi|>\sqrt{2}\}$. Thus F is an open n-dimensional simplex with totally geodesic faces and with one vertex (resp. two vertices) as cusp in case $n\leqq 8$ (resp. $n=9$). The fitness structure $(\mathcal{P},\mathcal{A})$ of F is defined as follows: Let $p_0^+, p_0^-, p_1,\cdots,p_{n-1}$ and p_n be respectively the points ${}^t(1,0,\cdots,0,1)$, ${}^t(0,1,0,\cdots,0,1)$, ∞, ${}^t(0,0,\cdots,0,\sqrt{2})$, ${}^t(1/2,1/2,0,\cdots,0,\sqrt{3/2})$, $1/2\,{}^t(1,1,1,\cdots,0,\sqrt{5}),\cdots,1/2\,{}^t(1,1,\cdots,1,0,\sqrt{9-(n-1)})$ and $1/2\,{}^t(1,1,\cdots,1,\sqrt{9-n})$ of $\boldsymbol{H}\cup\partial\boldsymbol{H}$. The $(n+1)$ points $p_0^+,p_0^-,p_1,p_2,p_4,\cdots,p_n$ are the vertices of F and p_3 is the middle point of p_0^+ and p_0^-. Let H_i^+ (resp. H_i^-) $(1\leqq i\leqq n)$ be the open $(n-1)$-simplex with vertices p_0^+ (resp. p_0^-), $p_1,\cdots,p_{i-1},p_{i+1},\cdots,p_n$. Clearly $\partial F=\bigcup_{i=1}^n(\bar H_i^+\cup\bar H_i^-)$. We denote by $\mathcal{P}$ the collection of simplices F, H_i^+, H_i^- $(1\leqq i\leqq n)$ and the lower dimensional face simplices of H_i^+ or H_i^- $(1\leqq i\leqq n)$. $\mathcal{P}$ is a subdivision of $\bar F$. We can find the elements $g_i\in\Gamma^\infty$ $(2\leqq i\leqq n)$ such that $g_iH_i^+=H_i^-$. Notice that $g_1H_1^+=H_1^-$. Let $\mathcal{A}$ be the subset $\{g_1,\cdots,g_n\}$ of Γ. Then

Proposition 1. *The pair $(\mathcal{P},\mathcal{A})$ is fit for F in the sense of* [1]. *$\mathcal{A}$ is a set of generators of $\Gamma^{(n)}$ $(4\leqq n\leqq 9)$ whose fundamental relations are $g_i^2=e$ $(1\leqq i\leqq n,\ i\neq 3)$, $g_3^3=e$, $(g_i^{-1}g_j)^2=e$ $(1\leqq i<i+2\leqq j\leqq n)$, $(g_i^{-1}g_{i+1})^3=e$ $(1\leqq i\leqq n-2)$ and $(g_{n-1}^{-1}g_n)^4=e$. In consequence $\Gamma/[\Gamma,\Gamma]=\boldsymbol{Z}/2\boldsymbol{Z}$ generated by g_n.*

Remark. In case $n=2$ (resp. $n=3$), $\Gamma/[\Gamma,\Gamma]=\boldsymbol{Z}/2\boldsymbol{Z}\oplus\boldsymbol{Z}/4\boldsymbol{Z}$ (resp. $(\boldsymbol{Z}/2\boldsymbol{Z})^2$). The group $\Gamma^{(n)}$ $(n\leqq 9)$ is a subgroup with index 2 of the group $G^{(n)}\cap GL(n+1,\boldsymbol{Z})$ which is generated by reflections. The latter group has been treated by Vinberg [2].

2°. We shall assume $n=3$. We define a polyhedron $F_{m,l}$ for each pair of integers $m\geqq 2$, $l\geqq 1$. By a sphere we shall mean an upper half sphere with centre in $\partial\boldsymbol{H}$. Let H_0^+ be a sphere with centre c. Let $H_{1,0}^+, H_{1,0}^-,\cdots,H_{m,0}^+,H_{m,0}^-$ be $2m$-spheres of the same radius such that each $H_{i,0}^+$ or $H_{i,0}^-$ $(1\leqq i\leqq m)$ crosses normally H_0^+ and that each $H_{i,0}^-$ (resp. $H_{i,0}^+$) is tangent to $H_{i,0}^+$ (resp. $H_{i-1,0}^-$) $(1\leqq i\leqq m)$ at a point of the circle ∂H_0^+, where $H_{0,0}^-$ means $H_{m,0}^-$. Assume that they are arranged counter-clockwise around H_0^+ in the above order. Let $K_{1,1}^+, L_{1,1}^+, K_{2,1}^+, L_{2,1}^+,\cdots,K_{m,1}^+,L_{m,1}^+$ be $2m$ spheres of the same radius and with centres outside H_0^+ such that



each $K_{i,1}^+$ (resp. $L_{i,1}^+$) crosses normally both $H_{i-1,0}^-$ and $H_{i,0}^+$ (resp. both $H_{i,0}^+$ and $H_{i,0}^-$) and is tangent to the three spheres $H_0^+, L_{i-1,1}^+, L_{i,1}^+$ (resp. $H_0^+, K_{i,1}^+, K_{i+1,1}^+$) (see the figure). They are uniquely determined and turn counterclockwise around H_0^+. Let V_1 (resp. W_1) be the sphere with centre c such that each $H_{i,0}^\varepsilon$ $(1\leqq i\leqq m,\ \varepsilon=\pm)$ (resp. each $K_{i,1}^+, L_{i,1}^+$ $(1\leqq i\leqq m)$) is tangent to V_1 (resp. W_1) from the inside of V_1 (resp. W_1). Let $H_{i,1}^\varepsilon$ $(1\leqq i\leqq m,\ \varepsilon=\pm)$ be the reflection of $H_{i,0}^\varepsilon$ with respect to V_1. Let $K_{i,1}^-$ (resp. $L_{i,1}^-$) $(1\leqq i\leqq m)$ be the reflection of $K_{i,1}^+$ (resp. $L_{i,1}^+$) with respect to W_1. Inductively we define V_j (resp. W_j) $(2\leqq j\leqq 2l)$ to be the reflection of V_{j-1} (resp. W_{j-1}) with respect to W_{j-1} (resp. V_j), $H_{i,j}^\varepsilon$ $(1\leqq i\leqq m,\ \varepsilon=\pm,\ 2\leqq j\leqq 2l)$ be the reflection of $H_{i,j-1}^\varepsilon$ with respect to V_j, $K_{i,j}^+$ (resp. $L_{i,j}^+$) $(1\leqq i\leqq m,\ 2\leqq j\leqq l)$ be the reflection of $K_{i,j-1}^-$ (resp. $L_{i,j-1}^-$) with respect to W_{2j-2} and $K_{i,j}^-$ (resp. $L_{i,j}^-$) $(1\leqq i\leqq m,\ 2\leqq j\leqq l)$ be the reflection of $K_{i,j}^+$ (resp. $L_{i,j}^+$) with respect to W_{2j-1}. Let H_0^- be the sphere W_{2l}. We denote by $F_{m,l}=F$ the polyhedron

$$\Big(\bigcap_{\substack{1\leqq i\leqq m\\ 0\leqq j\leqq 2l,\ \varepsilon=\pm}} D_{H_{i,j}^\varepsilon}\Big)\cap\Big(\bigcap_{\substack{1\leqq i\leqq m\\ 1\leqq j\leqq l,\ \varepsilon=\pm}} (D_{K_{i,j}^\varepsilon}\cap D_{L_{i,j}^\varepsilon})\Big)\cap D_{H_0^+}\cap D_{H_0^-}^*$$

with $2(4ml+m+1)-$ faces, where D_H (resp. D_H^*) means the half space over (resp. under) the half sphere H. We denote by $\mathscr{P}$ the naturally defined subdivision of $\bar{F}_{m,l}$. We denote each 2-dimensional face of F corresponding to the sphere $H_{i,j}^\varepsilon$, etc. by the same letter. Let $\alpha_{i,j}$ $(1\leqq i\leqq m,\ 0\leqq j\leqq 2l)$ be the element of G such that $\alpha_{i,j}H_{i,j}^+=H_{i,j}^-$, $\alpha_{i,j}F\cap F=\phi$ and $\alpha_{i,j}$ fixes the tangent point of $H_{i,j}^+$ and $H_{i,j}^-$. $\beta_{i,j}$ (resp. $\gamma_{i,j}$) $(1\leqq i\leqq m,\ 1\leqq j\leqq l)$ be the element of G such that $\beta_{i,j}K_{i,j}^+=K_{i,j}^-$ (resp. $\gamma_{i,j}L_{i,j}^+=L_{i,j}^-$), $\beta_{i,j}F\cap F=\phi$ (resp. $\gamma_{i,j}F\cap F=\phi$) and $\beta_{i,j}$ (resp. $\gamma_{i,j}$) fixes the tangent point of $K_{i,j}^+$ and $K_{i,j}^-$ (resp. $L_{i,j}^+$ and $L_{i,j}^-$). Let γ be the element of G such that $\gamma H_0^+=H_0^-$, $\gamma F\cap F=\phi$ and γ stabilizes any line through c in ∂H.

Proposition 2. *Let $\mathcal{A}$ be the collection of all the $\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j}, \gamma$ above. Then $(\mathcal{P}, \mathcal{A})$ is fit for $F_{m,l}$. The group $\Gamma_{m,l}$ generated by $\mathcal{A}$ has the fundamental relations $\gamma^{-1}\alpha_{i,0}^{-1}\gamma\alpha_{i,2l}=e$, $\beta_{i,j}\alpha_{i,2j-2}^{-1}\beta_{i+1,j}^{-1}\alpha_{i,2j}=e$, $\beta_{i,j}\alpha_{i,2j-1}^{-1}\beta_{i+1,j}^{-1}\alpha_{i,2j-1}=e$, $\gamma_{i,j}\alpha_{i,2j-2}^{-1}\gamma_{i,j}^{-1}\alpha_{i,2j}=e$ and $\gamma_{i,j}\alpha_{i,2j-1}^{-1}\gamma_{i,j}^{-1}\alpha_{i,2j-1}=e$ $(1 \leqq i \leqq m,\ 1 \leqq j \leqq l)$. In consequence $\Gamma_{ml}/[\Gamma_{ml}, \Gamma_{m,l}]=\boldsymbol{Z}^{2ml+m+l+1}$.*

Remark. $\Gamma_{m,l}$ is torsion free and the Betti numbers of $\boldsymbol{H}/\Gamma_{m,l}$ are $b_0=1$, $b_1=b_2=2ml+m+l+1$ and $b_3=0$. The volume of $\boldsymbol{H}/\Gamma_{m,l}$ equals to $8ml\left(2\varepsilon_0-\frac{1}{2}\int_0^{\pi/2m}\log\frac{1+\sin x}{1-\sin x}dx\right)$ where $\varepsilon_0=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)^2}$.

References

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[2] E. B. Vinberg: Discrete Groups Generated by Reflections in Lobacevskii Spaces. Math. USSR-Sbornik, Vol. 1, No. 3 (1967).