1. On Cauchy Problem for a System of Linear Partial Differential Equations with Constant Coefficients

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1. Introduction. We shall consider the Cauchy problem for a system of partial differential equations for a system of unknown functions $u_\mu = u_\mu(t, x)$ $(\mu=1, \cdots, k)$ of two independent real variables t and x:

$$\partial_t u_\mu = \sum_{\nu=1}^{k} P_{\mu\nu}(\partial_x) u_\nu \qquad (\mu=1, \cdots, k),$$

where $P_{\mu\nu}(\zeta)$ are polynomials in ζ with constant complex coefficients. Using vector-matrix notations we can write for the above system of equations as

$$(1) \qquad \partial_t u^{\downarrow} = \boldsymbol{P}(\partial_x) u^{\downarrow},$$

where $u^{\downarrow} = (u_\mu, \mu \downarrow 1, \cdots, k)$ and $\boldsymbol{P}(\zeta) = (P_{\mu\nu}(\zeta)_{\nu \to 1, \cdots, k}^{\mu \downarrow 1, \cdots, k})$.

Let $\mathcal{F}$ be a linear space of (generalized) complex vector valued functions on $\boldsymbol{R}^1$ such that $\mathcal{S}^k \subset \mathcal{F} \subset \mathcal{S}'^k$,[1] where the topology of the space on the left side of $\subset$ is finer than that of the space on the right side of $\subset$.

The Cauchy problem for the equation (1) is said to be forward $\mathcal{F}$-well posed on the interval $[0, \tau]$ $(\tau > 0)$, if and only if the following two conditions are satisfied.

1) (*Unique existence of the solution*) *For any* $u_0^{\downarrow} \in \mathcal{F}$ there exists a unique $\mathcal{F}$-valued solution $u^{\downarrow} = u^{\downarrow}(t, x)$ of (1) for $t \in [0, \tau]$ with the initial condition $u^{\downarrow}(0, x) = u_0^{\downarrow}(x)$.

2) (*Continuity of solution with respect to the initial value*) If the initial value $u_0^{\downarrow}$ tends to zero in $\mathcal{F}$, then the solution $u^{\downarrow} = u^{\downarrow}(t, x)$ of (1) with the initial value $u^{\downarrow}(0, x) = u_0^{\downarrow}(x)$ also tends to zero in $\mathcal{F}$ uniformly for $t \in [0, \tau]$.

Since the operator $P(\partial_x)$ does not depend on the time variable t, we can easily see that the forward $\mathcal{F}$-well posedness does not depend on $\tau > 0$, hence we can simply use the forward $\mathcal{F}$-well posedness without mentioning the interval $[0, \tau]$.

Making use of the Fourier transform with respect to the space variable x

$$v^{\downarrow}(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\xi x} u^{\downarrow}(x) dx,$$

1) $u^{\downarrow} \in \mathcal{S}^k$ $(\mathcal{S}'^k)$ means that $u_\mu \in \mathcal{S}$ $(\mathcal{S}')$ for every $\mu=1, \cdots, k$, where $\mathcal{S}$ denotes the set of all rapidly decreasing C^∞ functions on $\boldsymbol{R}^1$ and $\mathcal{S}'$ means the dual space of $\mathcal{S}$.

the Cauchy problem of the equation (1) can be formally reduced to that of the ordinary differential equation for the $\hat{\mathcal{F}}$-valued unknown function $v^{\downarrow}=v^{\downarrow}(t,\xi)$

(2) $$\partial_t v^{\downarrow}=P(i\xi)v^{\downarrow},$$

where $\hat{\mathcal{F}}$ is the Fourier transform of $\mathcal{F}$.

It is well known that for some function spaces, for example for $\mathcal{F}=\mathcal{S}^k$ or $(\mathcal{D}_{L^2})^k$, the necessary and sufficient condition for the forward $\mathcal{F}$-well posedness of (1) is given by the Petrovski correctness: "*The real parts of all eigen-values of the matrix* $\boldsymbol{P}(i\xi)$ are bounded above for $\xi\in\boldsymbol{R}^1$."[2]

In this note we shall show that the Petrovski correctness is *necessary* for the $\mathcal{F}$-well posedness of (1) provided that $\mathcal{S}^k\subset\mathcal{F}\subset\mathcal{S}'^k$.

2. The necessity of the Petrovski correctness. In the case $\mathcal{S}^k\subset\mathcal{F}\subset\mathcal{S}'^k$, the necessity of the Petrovski correctness for the forward $\mathcal{F}$-well posedness comes from the following proposition.

Proposition. *If $\boldsymbol{P}(i\xi)$ does not satisfy the Petrovski correctness, then, for the solution $v^{\downarrow}=v^{\downarrow}(t,\xi)$ of the equation* (2), *we can construct a sequence of initial values $v_n^{\downarrow}(\xi)\subset C_0^\infty(\boldsymbol{R}^1)$*[3] *such that $v_n^{\downarrow}\to 0$ in $\mathcal{S}^k$ as $n\to\infty$, but, at $t=\tau>0$, $v_n^{\downarrow}(\tau,\xi)\not\to 0$ in $\mathcal{S}'^k$ as $n\to\infty$.*

To prove this proposition, let $\lambda=\tilde{\lambda}(\xi)$ be eigen-value of $\boldsymbol{P}(i\xi)$ such that

$$\mathcal{R}e\,\tilde{\lambda}(\xi)=\mathrm{Max}\,\{\mathcal{R}e\,\lambda_j(\xi)\,;\,j=1,\cdots,k\}.$$

And we use following lemmas, of which we shall omit the proof.

Lemma 1. *There exist $l\in\boldsymbol{N}$ and $h\in\boldsymbol{Z}$*[4] *and a normalized*[5] *eigen-vector $v_0^{\downarrow}(\xi)$ of $\boldsymbol{P}(i\xi)$ corresponding to the eigen-value $\tilde{\lambda}(\xi)$ such that, for $\xi\geqq R$ with a sufficiently large $R>0$,*

$$\tilde{\lambda}(\xi)=\xi^{h/l}f(\xi^{-1/l}),\qquad v_\nu(\xi)=f_\nu(\xi^{-1/l}),$$

$(v_0^{\downarrow}(\xi))=(v_1(\xi),\cdots,v_k(\xi))$, *where $f(\zeta)$ and $f_\nu(\zeta)$ are regular analytic for $|\zeta|\leqq R^{-1}$ and $f(0)\neq 0$.*

Lemma 2. *Let $\varepsilon>0$ and $\rho\in C_0^\infty(\boldsymbol{R}^1)$ be such that*

$$\mathrm{supp}\,(\rho)\subset[-1,1],\quad \rho(\xi)\geqq 0,\quad \int_{-1}^{1}\rho(\xi)d\xi=1,$$

and let

$$v_{(\alpha)}^{\downarrow}(\xi)=\exp\,(-2^{-1}(1+\xi^2)^{\varepsilon})\rho(\xi-\alpha)v_0^{\downarrow}(\xi),$$

where $v_0^{\downarrow}(\xi)$ is the eigen-vector of $\boldsymbol{P}(i\xi)$ given in Lemma 1.

Then, there exists $R_1>R>0$ such that $v_{(\alpha)}^{\downarrow}\subset\mathcal{S}^k$ for $\alpha\geqq R_1$ and $v_{(\alpha)}^{\downarrow}\to 0$ in $\mathcal{S}^k$ as $\alpha\to+\infty$.

Lemma 3. *Let $\tilde{\lambda}(\xi)$ and $v_0^{\downarrow}(\xi)$ be the same as above, and $\psi(\xi)\geqq 0$*

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2) Cf. [1] and [2] of the references.

3) By $C_0^\infty(\boldsymbol{R}^1)$ we denote the set of all complex valued C^∞ functions on $\boldsymbol{R}^1$ with compact support.

4) N=the set of all natural numbers. Z=the set of all rational integers.

5) $|v^{\downarrow}|=(\sum_{\nu=1}^{k}|v_\nu|^2)^{1/2}=1$, if $v^{\downarrow}=(v_1,\cdots,v_k)$.

be a C^∞ function such that $\psi(\xi)=0$ for $\xi\leqq R_1$ and $\psi(\xi)=1$ for $\xi\geqq R_1+1$. Then, for any $\varepsilon>0$ and $\tau>0$,

$$\phi^1(\xi)=\psi(\xi)\exp(-2^{-1}(1+\xi^2)^\varepsilon-i\tau\mathcal{I}_m\tilde{\lambda}(\xi))\bar{v}_0^1(\xi)^{6)}\in\mathcal{S}^k.$$

Proof of Proposition. Let $v^1_{(\alpha)}(\xi)$ be the vector given in Lemma 2, which is also an eigen-vector of $\boldsymbol{P}(i\xi)$ corresponding to the eigen-value $\tilde{\lambda}(\xi)$, and put

$$v^1_{(\alpha)}(t,\xi)=\exp(t\tilde{\lambda}(\xi))v^1_{(\alpha)}(\xi).$$

Then $v^1=v^1_{(\alpha)}(t,\xi)$ $(t\geqq 0)$ is the solution of the equation (2) with the initial condition $v^1_{(\alpha)}(0,\xi)=v^1_{(\alpha)}(\xi)$. By Lemma 2 we have $v^1_{(\alpha)}\to 0$ in $\mathcal{S}^k$ as $\alpha\to+\infty$. Now assume that $\mathcal{R}_e\,\tilde{\lambda}(\xi)$ is *not* bounded above for $0\leqq\xi<\infty$.[7] Then, by Lemma 1, we have $h\geqq 1$ and $\mathcal{R}_e\,f(0)=a>0$. Let $\phi^1(\xi)$ be the function given in Lemma 3. Then, if $\alpha>R_1+1$, we have

$$\begin{aligned}\langle v^1_{(\alpha)}(\tau,\cdot),\phi^1(\cdot)\rangle_{R^1} &=\int_{-\infty}^{\infty}\psi(\xi)\rho(\xi-\alpha)\exp(\tau\,\mathrm{Re}\,\tilde{\lambda}(\xi)-(1+\xi^2)^\varepsilon)d\xi\\ &=\int_{\alpha-1}^{\alpha+1}\rho(\xi-\alpha)\exp(\tau\,\mathrm{Re}\,\tilde{\lambda}(\xi)-(1+\xi^2)^\varepsilon)d\xi.\end{aligned}$$

And, as

$$\int_{\alpha-1}^{\alpha+1}\rho(\xi-\alpha)d\xi=1,$$

by mean value theorem,

$$\langle v^1_{(\alpha)}(\tau,\cdot),\phi^1(\cdot)\rangle=\exp(\tau\,\mathrm{Re}\,\tilde{\lambda}(\xi_1)-(1+\xi_1^2)^\varepsilon)$$

with some $\xi_1\in(\alpha-1,\alpha+1)$. But, as $\mathrm{Re}\,\tilde{\lambda}(\xi)=\xi^{h/l}(a+\delta(\xi))$, where $\delta(\xi)\to 0$ as $\xi\to\infty$, taking ε such that $0<\varepsilon<h/(2l)$, we have, as $\alpha\to\infty$,

$$\tau\,\mathrm{Re}\,\tilde{\lambda}(\xi_1)-(1+\xi_1^2)^\varepsilon=\xi_1^{h/l}(\tau a+\delta_1(\xi_1))\to+\infty,$$

where $\delta_1(\xi_1)\to 0$ as $\xi_1\to\infty$. This shows that $v^1_{(\alpha)}(\tau,\cdot)\not\to 0$ in $\mathcal{S}'^k$ as $\alpha\to+\infty$. Q.E.D.

As the Fourier transform is an isomorphic and homeomorphic mapping of $\mathcal{S}$ onto $\mathcal{S}$ and of $\mathcal{S}'$ onto $\mathcal{S}'$, we obtain, in consequence of Proposition, the following theorem.

Theorem. *Let $\mathcal{S}^k\subset\mathcal{F}\subset\mathcal{S}'^k$, where the topology of the space on the left side of $\subset$ is finer than that of on the right side of $\subset$. Then the Petrovski correctness is necessary for the Cauchy problem of the equation* (1) *to be forward $\mathcal{F}$-well posed.*

References

[1] S. Mizohata: The Theory of Partial Differential Equations. Cambridge Univ. Press (1973).

[2] I. M. Gelfand and G. E. Shilov: Generalized Functions, Vol. 3 (Translated from Russian) (1967). Academic Press.

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6) $\bar{v}^t=(\bar{v}_1,\cdots,\bar{v}_k)$ if $v^t=(v_1,\cdots,v_k)$, hence $(v^t,\bar{v}^t)=|v|^2$.

7) The proof goes quite similarly, when $\mathcal{R}_e\,\tilde{\lambda}(\xi)$ is not bounded above for $-\infty<\xi\leqq 0$.