36. On the Existence of Invariant Functions for Markov Representations of Amenable Semigroups

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§ 0. **Introduction.** Let (X, Σ, m) be a σ-finite measure space and S be a left amenable semigroup. By L^1 and L^∞ we denote the usual Banach spaces $L^1(X, \Sigma, m)$ and $L^\infty(X, \Sigma, m)$ respectively. Let $\boldsymbol{T} =\{T_s; s \in S\}$ be a representation of S by positive linear contractions on L^1. For the sake of brevity such $\boldsymbol{T}$ is called a *Markov representation* of S on L^1. By $co(\boldsymbol{T})$ we denote the convex hull of $\{T_s; s \in S\}$ and by $\overline{co}(\boldsymbol{T})$ the closure of $co(\boldsymbol{T})$ with respect to the operator norm topology. For this $\boldsymbol{T}$ we consider the following conditions:

(A) *There exists a strictly positive function f in L^1 such that $T_sf =f$ for all $s \in S$.*

(B) *Every operator in $\overline{co}(\boldsymbol{T})$ is conservative.*

(C) *T_s is conservative for every $s \in S$.*

Then it is obvious that the condition (A) implies (B) and (C). In this paper we shall prove the next theorems.

Theorem 1. *For any Markov representation $\boldsymbol{T}=\{T_s; s \in S\}$ of a left amenable semigroup S on L^1, the conditions* (A) *and* (B) *are mutually equivalent.*

Theorem 2. *Let S be an extremely left amenable semigroup. Then for any Markov representation $\boldsymbol{T}=\{T_s; s \in S\}$ of S on L^1 with the following property*:

(1) $T_s^*(gh)=T_s^*(g)T_s^*(h)$ *for any $g, h \in L^\infty$ and $s \in S$,*

the conditions (A) *and* (C) *are mutually equivalent.*

Theorem 1 is proved by Brunel [1] for the case when S is the additive semigroup of positive integers, and by Horowitz [3] for the case when S is commutative. In the author's paper [4] we shall show that the main theorem in [3] is also valid for the case of left amenable semigroups of Markov operators.

§ 1. **Proof of Theorem 1.** Let S and $\boldsymbol{T}=\{T_s; s \in S\}$ be as in Theorem 1. By $L(\boldsymbol{T})$ we denote the closed linear subspace of L^∞ generated by $\{T_s^*h-h; s \in S, h \in L^\infty\}$ and put $L^+(\boldsymbol{T})=\{h \in L(\boldsymbol{T}); h \geq 0\}$. Then the next lemma is well-known (e.g., see Granirer [2, Theorem 5]).

Lemma 3. *For any $f \in L^\infty$ the following equality holds*:

(2) $\inf \{\|f-h\|_\infty; h \in L(\boldsymbol{T})\}=\inf \{\|Q^*f\|_\infty; Q \in co(\boldsymbol{T})\}$.

Especially if S is extremely left amenable, then

(3) $\inf\{\|f-h\|_\infty;\ h \in L(T)\} = \inf\{\|T_s^* f\|_\infty;\ s \in S\}$.

Combining with Lemma 3 and Theorem 1(5) in Takahashi [5], we have

Lemma 4. *For any Markov representation $T=\{T_s;\ s \in S\}$ of S on L^1, the condition* (A) *holds if and only if $L^+(T)=\{0\}$.*

The next lemma is essential for us to prove Theorem 1.

Lemma 5. *If $h \in L^+(T)$, then there exists a $V \in \overline{co}(T)$ such that $\lim_{n\to\infty} \|V^{*n}h\|_\infty = 0$.*

Proof. Let $\{\alpha_n;\ n=1,2,\cdots\}$ be a sequence of positive numbers with $\sum_{i=1}^\infty \alpha_i = 1$, and put $\beta_n = \sum_{i=1}^n \alpha_i$ and $\tilde\beta_n = 1-\beta_n$. Moreover we choose an increasing sequence $\{\gamma_n\}$ of positive integers satisfying $\lim_{n\to\infty} \beta_n^{\gamma_n} = 0$. We can take a sequence $\{Q_n\}$ in $co(T)$ such that

(4) $$\|Q_n^* h_n\|_\infty < \frac{1}{n} \qquad \text{for } n=1,2,\cdots,$$

where $h_1 = h$, $h_n = h + \sum_{i=1}^{n-1} \sum_{k=0}^{\gamma_i} V_i^{*k} h$ for $n \geq 2$, and $V_i = \beta_i^{-1} \sum_{j=1}^i \alpha_j Q_j$. Indeed, since $h_n \in L^+(T)$ for all $n \geq 1$, we have $\inf\{\|Q^* h_n\|_\infty;\ Q \in co(T)\} = 0$ by (2). So the desired sequence $\{Q_n\}$ can be taken inductively. We now put $V = \sum_{i=1}^\infty \alpha_i Q_i$ and $\tilde V_n = \tilde\beta_n^{-1} \sum_{i=n+1}^\infty \alpha_i Q_i$. Then $V \in \overline{co}(T)$, $V = \beta_n V_n + \tilde\beta_n \tilde V_n$, and from (4) we have

(5) $$\|\tilde V_n^* V_n^{*k} h\|_\infty < \frac{1}{n+1} \qquad \text{for all } n \geq 1 \text{ and } 0 \leq k \leq \gamma_n.$$

For any given $\varepsilon > 0$ we can find a positive integer n such that $\beta_n^{\gamma_n} \|h\|_\infty < \varepsilon/2$ and $(n+1)^{-1} < \varepsilon/2$. Putting $N = \gamma_n$, by (5) we have

$$\|V^{*N}h\|_\infty = \|(\beta_n V_n^* + \tilde\beta_n \tilde V_n^*)^N h\|_\infty < \beta_n^N \|V_n^{*N} h\|_\infty + \frac{1}{n+1}(1-\beta_n^N) < \varepsilon.$$

So $\|V^{*k}h\|_\infty < \varepsilon$ for all $k \geq N$. Hence this V has our desired property.

q.e.d.

Using Lemma 5, we can prove the following lemma by the same method as in Theorem 1 in [1].

Lemma 6. *For any $h \in L^+(T)$ there exists a $U \in \overline{co}(T)$ such that $\sum_{k=0}^\infty U^{*k} h \in L^\infty$.*

From Lemma 6 it follows that if $L^+(T)$ contains a non-zero function, then in $\overline{co}(T)$ there exists at least one operator which is not conservative. Hence if the condition (B) holds, then $L^+(T)=\{0\}$. Owing to Lemma 4, we can conclude that the condition (B) implies (A) for any Markov representation of S on L^1. Thus Theorem 1 is proved completely.

§2. Proof of Theorem 2. Let S and $T=\{T_s;\ s \in S\}$ be as in Theorem 2. Suppose now that $L^+(T)$ contains a non-zero function. Then there exists an $A \in \Sigma$, $m(A) > 0$ such that the indicator function $h = I_A$ of A belongs to $L^+(T)$. By (3) we can take an element $s \in S$ satisfying $\|T_s^* h\|_\infty < 1$. Since $h = h^2$, we have $\|T_s^* h\|_\infty = \|(T_s^* h)^2\|_\infty \leq \|T_s^* h\|_\infty^2$.

So $\|T_s^* h\|_\infty = 0$. This means that T_s is not conservative. Hence recalling Lemma 4, we conclude that the condition (C) implies (A) for any Markov representation of S on L^1 with (1). Thus Theorem 2 is proved completely.

References

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