35. On Certain Ray Class Invariants of Real Quadratic fields

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1. In this article we introduce certain ray class invariants of real quadratic fields which are intimately related to the values at $s=1$ of certain L functions of the fields. Then we present a few conjectures on the arithmetic nature of the invariants. Our conjectures are closely related to the third conjecture presented by H. M. Stark in his Nice Congress talk [5]. Assuming suitable additional hypotheses, we prove the conjecture. Unfortunately, the hypotheses are rather restrictive. However, there are several numerical evidences which are in favor of the conjecture even when the hypotheses are not satisfied. The full exposition of the present paper will appear elsewhere.

2. For a pair of *positive* numbers $\omega=(\omega_1, \omega_2)$, we denote by $\Gamma_2(z, \omega)$ the double gamma function introduced and studied by E. W. Barnes in [1]. If we use notations of [3], $\Gamma_2(z, \omega)=F(\omega, z)^{-1}$. Set

$$\Phi(z, \omega)=\frac{\Gamma_2(z, \omega)}{\Gamma_2(\omega_1+\omega_2-z, \omega)}.$$

If ω_1/ω_2 *is irrational,* $\Phi(z, \omega)$ is, as a function of z, characterized by the following properties (1) and (2).

(1) $\Phi(z, \omega)$ is a meromorphic function of z which satisfies the following difference equations:

$$\Phi(z+\omega_1, \omega)=2 \sin\left(\frac{\pi z}{\omega_2}\right)\Phi(z, \omega),$$

$$\Phi(z+\omega_2, \omega)=2 \sin\left(\frac{\pi z}{\omega_1}\right)\Phi(z, \omega).$$

(2) $\Phi\left(\frac{\omega_1+\omega_2}{2}, \omega\right)=1.$

3. Let F be a real quadratic field embedded in the real field $\boldsymbol{R}$. For an integral ideal $\mathfrak{f}$ of F, we denote by $H(\mathfrak{f})$ the group of *narrow* ray classes modulo $\mathfrak{f}$. *We assume that* $\mathfrak{f}$ *satisfies the following conditions* (3) *and* (4).

(3) For any totally positive unit u of F, $u+1 \notin \mathfrak{f}$.

(4) There is no unit u of F such that $u-1 \in \mathfrak{f}$, $u>0$ and $u'<0$, where u' is the conjugate of u.

Take a totally positive integer ν of F such that $\nu+1 \in \mathfrak{f}$. We denote by the same letter ν the ray class modulo $\mathfrak{f}$ represented by the principal ideal (ν). Furthermore, choose an integer μ of F such that $\mu<0$, $\mu'>0$ and $\mu-1\in\mathfrak{f}$. We denote by the same letter μ the ray class represented by (μ). The condition (3) implies that ν is an element of order 2 of the group $H(\mathfrak{f})$. On the other hand, μ is an element of order at most 2 of $H(\mathfrak{f})$.

Choose integral ideals $\mathfrak{a}_1, \cdots, \mathfrak{a}_{h_0}$ of F so that they form a complete set of representatives for *narrow* ideal classes of F. For each $c\in H(\mathfrak{f})$, there exists a unique index j $(1\leqslant j\leqslant h_0)$ such that c and $\mathfrak{a}_j\mathfrak{f}$ are in the same *narrow* ideal class of F. Let $\varepsilon>1$ be the fundamental totally positive unit of F and put

$$R_j(\mathfrak{f})=\{z=x+y\varepsilon\in(\mathfrak{a}_j\mathfrak{f})^{-1};\ x, y\in \boldsymbol{Q}\qquad 0<x\leqslant 1,\ 0\leqslant y<1\}.$$

Then $R_j(\mathfrak{f})$ is a *finite* subset of the fractional ideal $(\mathfrak{a}_j\mathfrak{f})^{-1}$. Set

$$X_{\mathfrak{f}}(c)=\prod_z \Phi(z, (1, \varepsilon))\Phi(z', (1, \varepsilon')),$$

where the product is over all $z\in R_j(\mathfrak{f})$ such that $(z)\mathfrak{a}_j\mathfrak{f}=c$ in $H(\mathfrak{f})$. Then $X_{\mathfrak{f}}(c)$ is a positive real number.

For each $c\in H(\mathfrak{f})$, set $\zeta_F(s, c)=\sum N(\mathfrak{g})^{-s}$, where the summation is over all integral ideals of F which are in the same narrow ray class modulo $\mathfrak{f}$ as c. It is well-known that $\zeta_F(s, c)$ is a meromorphic function of s which is holomorphic except for a simple pole at $s=1$. The following result, which is implicit in Corollary 2 to Theorem 1 of [4], guarantees that $X_{\mathfrak{f}}(c)$ is independent of the choice of $\mathfrak{a}_1, \cdots, \mathfrak{a}_{h_0}$.

Theorem 1. *The notation being as above,*

$$\zeta'_F(0, c)-\zeta'_F(0, c\nu)=\log X_{\mathfrak{f}}(c).$$

Let G be a subgroup of $H(\mathfrak{f})$. *Assume that μ is in G but ν is not in G.* Set

$$X_{\mathfrak{f}}(c, G)=\prod_{g\in G} X_{\mathfrak{f}}(cg).$$

Then $X_{\mathfrak{f}}(c, G)$ is an invariant for $c\in H(\mathfrak{f})/G$. Denote by $K(\mathfrak{f})$ the ray class field over F with conductor $(\infty)(\infty)'\mathfrak{f}$. Denote by σ the Artin canonical isomorphism from the group $H(\mathfrak{f})$ onto the Galois group of $K(\mathfrak{f})$ with respect to F. Let $K(\mathfrak{f}, G)$ be the subfield of $\sigma(G)$-fixed elements of $K(\mathfrak{f})$. Now we present the following conjecture.

Conjecture. *There exists a positive rational integer m such that the following assertions* (i), (ii) *and* (iii) *hold*:

(i) The invariant $X_{\mathfrak{f}}(c, G)^m$ is a unit in the field $K(\mathfrak{f}, G)$. Moreover

$$\{X_{\mathfrak{f}}(c, G)^m\}^{\sigma(c_0)}=X_{\mathfrak{f}}(cc_0, G)^m \qquad (\forall c_0\in H(\mathfrak{f})).$$

(ii) The system of invariants $\bigcup_{\mathfrak{f}_0}\{X_{\mathfrak{f}_0}(c, \tilde{G})^m;\ c\in H(\mathfrak{f}_0)\}$,

where $\mathfrak{f}_0$'s are divisors of $\mathfrak{f}$ with the properties (3) and (4) and $\tilde{G}$ is the image of G under the natural homomorphism from $H(\mathfrak{f})$ onto $H(\mathfrak{f}_0)$, generates the field $K(\mathfrak{f}, G)$ over F.

(iii) Let τ be an embedding of the field $K(\mathfrak{f}, G)$ into the complex number field C which induces the non-trivial automorphism on F. Then $\tau(X_{\mathfrak{f}}(c, G)^m)$ is a complex number of modulus 1.

4. In this paragraph we assume that $\mathfrak{f}$ is invariant under the non-trivial automorphism $'$ of F. Then $'$ acts on the group $H(\mathfrak{f})$ in a natural manner. We further assume that there exists a subgroup G_1 of G which satisfies the following conditions (5) and (6).

(5) $G_1'=G_1$.

(6) The group G is generated by G_1 and by μ.

Set $(H(\mathfrak{f})/G_1)_0=\{c\in H(\mathfrak{f})/G_1;\ c'=c\}$.

Under our assumptions, it is easy to see that $\mu \notin (H(\mathfrak{f})/G_1)_0$.

Hence, $[H(\mathfrak{f})/G_1,\ (H(\mathfrak{f})/G_1)_0]\geqq 2$.

Theorem 2. *If* $[H(\mathfrak{f})/G_1;\ (H(\mathfrak{f})/G_1)_0]=2$, *the conjecture is true.*

Proof. Denote by K the subfield of $\sigma(G_1)$-fixed elements of $K(\mathfrak{f})$. Let L be the subfield of $\sigma((H(\mathfrak{f})/G_1)_0)$-fixed elements of K. Then L is a composition of F with a suitable imaginary quadratic field k. Moreover, K is abelian over k. Hence, K is a class field over k with conductor $\mathfrak{c}$. Let χ be a character of the group $H(\mathfrak{f})/G$ which satisfies $\chi(\nu)=-1$. Then $\chi\neq\chi'$. Let $\mathfrak{f}_\chi$ be the conductor of χ and let $\tilde{\chi}$ be the character of the group $H(\mathfrak{f}_\chi)$ which corresponds to χ in a natural manner. Identify χ with the character of Gal (K/F) and denote by ψ_χ the irreducible character of Gal $(K/\boldsymbol{Q})$ induced from χ. Then we have

$$L(s, \psi_\chi, K/\boldsymbol{Q})=L_F(s, \tilde{\chi}),$$

where $L(s, \psi_\chi, K/\boldsymbol{Q})$ is the Artin L-function of K associated with ψ_χ and $L_F(s, \tilde{\chi})$ is the L-function of F associated with the character $\tilde{\chi}$. On the other hand, the restriction of ψ_χ to Gal (K/k) is a direct sum of a character ξ_χ and its conjugate. Via the Artin reciprocity law, ξ_χ is identified with a primitive character of the group of ideal classes modulo $\mathfrak{c}_0$ of k, where $\mathfrak{c}_0$ is a suitable divisor of $\mathfrak{c}$. Furthermore, $L(s, \psi_\chi, K/\boldsymbol{Q})=L_k(s, \xi_\chi)$, where $L_k(s, \xi_\chi)$ is the L-function of k associated with the character ξ_χ. Thus we have

(7) $L_F(s, \tilde{\chi})=L_k(s, \xi_\chi)$.

Applying Theorem 1 and the equality (7) and results of Ramachandra [2], we can express $X_{\mathfrak{f}}(c, G)$ in terms of singular values of elliptic modular functions. Another application of Ramachandra's results now leads to our conjecture under our present hypotheses.

Remark. The group $(H(\mathfrak{f})/G_1)_0$ is a subgroup of index two of $H(\mathfrak{f})/G_1$ if and only if there is an imaginary quadratic field in K over which K is abelian.

5. Set $F=\boldsymbol{Q}(\sqrt{29})$, $\mathfrak{f}=((3-\sqrt{29})/2)$. Then the group $H(\mathfrak{f})$ is isomorphic to a cyclic group of order 4 generated by $c=(2)$. Set

$$Y_m=\sum_{i=0}^{3} X_{\mathfrak{f}}(c^i)^m.$$

Theorem 2 is not applicable for this example. However, if the conjecture were true, Y_m would be an integer in F whose conjugate is in the interval $(-4, 4)$. A numerical computation, which involves a computer machine, shows that the integral part of $Y_1/\sqrt{29}$ is 1 and that the difference between 4 and $Y_1-(1+\sqrt{29})/2$ is less than $(10)^{-25}$. Another numerical computation shows that the integral part of $Y_2/\sqrt{29}$ is 4 and that the integral part of $Y_2-4\cdot(1+\sqrt{29})/2$ is 12. Moreover, the difference between the fractional part of $Y_2-4(1+\sqrt{29})/2$ and the fractional part of $(1+\sqrt{29})/2$ is less than $(10)^{-24}$. Thus it is quite probable that the conjecture will be true for this example with $m=1$ and that $X_{\mathfrak{f}}(c^i)$ $(i=0,1,2,3)$ are roots of the following equation:

$$x^4-(9+\sqrt{29})/2\cdot x^3+(8+\sqrt{29})x^2-(9+\sqrt{29})/2\cdot x+1=0$$

Further numerical computation makes the validity of the following equalities quite probable.

$$X_{\mathfrak{f}}(c^0)=(t_1+\sqrt{t_1^2-4})/2, \qquad X_{\mathfrak{f}}(c)=(t_2+\sqrt{t_2^2-4})/2,$$
$$X_{\mathfrak{f}}(c^2)=(t_1-\sqrt{t_1^2-4})/2, \qquad X_{\mathfrak{f}}(c^3)=(t_2-\sqrt{t_2^2-4})/2,$$

where we put

$$t_1=((9+\sqrt{29})/2+\sqrt{(7+\sqrt{29})/2})/2,$$
$$t_2=(9+\sqrt{29})/2-t_1.$$

Added in proof. After the manuscript had been sent to the printer, the following paper came into the author's attention: H.M. Stark: L-functions at $s=1$. III. Totally real fields and Hilbert's twelfth problem. Adv. Math., **22**, 64–84 (1976). In the paper, a substantial part of the author's conjecture had been formulated in a more general context. However, both Theorems 1 and 2 of the present paper seem to remain to be new.

References

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