31. A Remark on the Dimensional Fullvaluedness

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1. Let $\mathcal{P}$ be the class of paracompact Hausdorff spaces. In this note we investigate a dimensionally fullvalued compactum for $\mathcal{P}$, that is, a compactum X such that $\dim(X\times Y)=\dim X+\dim Y$ for every $Y\in\mathcal{P}$. In particular, we shall show that every movable curve is dimensionally fullvalued for $\mathcal{P}$. This gives a partial answer to [8, Problem 9] or [14, Problem 42-5]. Throughout the paper every group is abelian.

2. Using the same argument as in the proof of [6, Lemma 2] the following is a consequence of [2, Theorem C].

Proposition 1. *Let G be a group and X a compactum with* $\dim X<\infty$ *such that $\check{H}^n(X, A:G)$ contains a copy of G as a direct summand for some closed G_δ set A of X. Then for every $Y\in\mathcal{P}$ with* $\dim Y<\infty$ *$D(X\times Y:G)\leqq n+D(Y:G)$. Here $\check{H}^*$ is the Čech cohomology and $D(X:G)$ is the cohomological dimension of X with respect to G* (cf. [14]).

A space X is said to be *G-cyclic* if $\check{H}^n(X:G)\neq 0$ for $n=\dim X$.

Theorem 1. *Let Q be a divisible abelian group and let X be a Q-cyclic compact metric space which is movable in the sense of Borsuk* [3]. *Then for every space $Y\in\mathcal{P}$ with* $\dim Y<\infty$ *and for every group G the relation $D(X\times Y:G)\geqq \dim X+D(Y:G)$ holds.*

Proof. By [15, 6. 8. 11] we have $\check{H}^n(X:Z)\otimes Q=\check{H}^n(X:Q)\neq 0$, where Z is the additive group of integers. Since Q is divisible, $\check{H}^n(X:Z)/\mathrm{Tor}\,(\check{H}^n(X:Z))\neq 0$, where $\mathrm{Tor}\,(H)$ is the torsion subgroup of H. By [5, Theorem 4.4], we can know that $\check{H}^n(X:Z)/\mathrm{Tor}\,(\check{H}^n(X:Z))$ has property L in the sense of Pontrjagin. Hence by [4, 13.1] $\check{H}^n(X:Z)/\mathrm{Tor}\,(\check{H}^n(X:Z))$ is a free abelian group, because $\check{H}^n(X:Z)$ is countable. Therefore Z is a direct summand of $\check{H}^n(X:Z)$. Since $\check{H}^n(X:G)=\check{H}^n(X:Z)\otimes G$, $\check{H}^n(X:G)$ contains a copy of G as a direct summand. The theorem follows from Proposition 1.

Corollary 1. *If Q is a divisible group, then every Q-cyclic movable compact metric space X is dimensionally fullvalued for $\mathcal{P}$.*

From [14, Theorems 40–7 and 40–8] it is known that the movability of X can not be omitted in Theorem 1 and Corollary 1.

Theorem 2. *Let X be a compact space such that* $0<\dim X<\infty$ *and $\check{H}^1(X:Z)=0$. Then for every group G and for every space $Y\in\mathcal{P}$ with* $\dim Y<\infty$ *the relation $D(X\times Y:G)\geqq D(Y:G)+1$ holds.*

Proof. There is a component X_0 of X with $\dim X_0>0$. Take two closed G_δ sets A and B of X_0 such that $A\neq\emptyset\neq B$ and $A\cap B=\emptyset$. Since $\check{H}^1(X_0:Z)=0$, we can see that $\check{H}^1(X_0, A\cup B:Z)$ is a free abelian group. Hence $\check{H}^1(X_0, A\cup B:G)$ contains a copy of G as a direct summand, because $\check{H}^1(X_0, A\cup B:Z)\otimes G$ is a direct summand of $\check{H}^1(X_0, A\cup B:G)$ by [15, 8.6.11]. The theorem follows from Proposition 1.

Corollary 2. *Let X be a compact space such that* $\dim X\geqq 1$ *and* $\check{H}^1(X:Z)=0$. *Then for every* $Y\in\mathcal{P}$ $\dim(X\times Y)\geqq\dim Y+1$.

Corollary 3. *If X is a compact space with* $\dim X\geqq 1$ *such that X is shape-1-connected* ([12, p. 61]), *in particular, if X is a non degenerate* ASR ([11]), *then for every* $Y\in\mathcal{P}$ $\dim(X\times Y)\geqq\dim Y+1$.

Corollary 4. *Let π be a class of connected finite polyhedra such that for every $P\in\pi$ $\check{H}^1(P:Z)=0$. If X is a π-like compact space* ([9]) *and* $1\leqq\dim X<\infty$, *then for every group G and for every $Y\in\mathcal{P}$ with* $\dim Y<\infty$ $D(X\times Y:G)\geqq D(Y:G)+1$.

From Corollary 4 it follows that if X is an S^2-like compact space with $\dim X\geqq 1$, where S^2 is a 2-sphere, then $D(X\times Y:G)\geqq D(Y:G)+1$ for each $Y\in\mathcal{P}$, $\dim Y<\infty$. If such a space X would contain an arc, then this is obvious (cf. [14, 42-3]). However it is not generally true. Under the continuum hypothesis Mardešić [8] constructed an S^2-like compact space X such that X is locally connected and contains no locally connected proper subcontinuum.

Theorem 3. *Every 1-dimensional movable compact metric space is dimensionally fullvalued for $\mathcal{P}$.*

The proof is obvious from Theorems 1 and 2.

Since every chainable continuum is of trivial shape, we have

Corollary 5. *Every chainable continuum is dimensionally fullvalued for $\mathcal{P}$.*

The movability of X in Theorem 3 is not necessary. For, consider a non-movable solenoid S. Since S is a Δ-space, by [7, Theorem 2] S is dimensionally fullvalued for $\mathcal{P}$.

The following theorem generalizes [13, Theorem 6].

Theorem 4. *Let G be a group and let Y be a space in $\mathcal{P}$ such that* $\dim Y<\infty$ *and* $D(Y:G)\geqq 1$. *Then for every compact space X with* $\dim X<\infty$ *the relation* $D(X\times Y:G)\geqq D(X:G)+1$ *holds.*

For the proof we need the following proposition.

Proposition 2. *Let Y be a space in $\mathcal{P}$. Then* $D(Y:G)\geqq 1$ *for some group G if and only if there exists a closed set F of Y such that* $\check{H}^1(Y,F:H)\neq 0$ *for any group H.*

Proof. The if part is trivial. Suppose that $D(Y:G)\geqq 1$ for some group G. There exist closed sets A and B of Y such that $A\neq\emptyset\neq B$, $A\cap B=\emptyset$ and $\overline{U}-U\neq\emptyset$ for any open set $U, A\subset U\subset\overline{U}\subset Y-B$. For a given group H, consider the exact sequence: $\to \check{H}^0(Y:H) \xrightarrow{i^*} \check{H}^0(A \cup B:H) \to \check{H}^1(Y, A\cup B:H) \to \cdots$. Here i is the inclusion of $A\cup B$ into Y. Since i^* is not onto, we have $\check{H}^1(Y, A\cup B:H)\neq 0$.

Proof of Theorem 4. Let $D(X:G)=n$. We can assume that $n>0$. Take a closed G_δ set A such that $\check{H}^n(X, A:G)\neq 0$. By X_A denote the quotient space X/A. Then $\check{H}^n(X_A:G)\neq 0$, because $n>0$. Since $D(X_A\times Y:G)\leqq D(X\times Y:G)$ by [14, 38-3], it is enough to prove $D(X_A\times Y:G)\geqq n+1$. By Proposition 2, there exists a closed set F of Y such that $\check{H}^1(Y, F:H)\neq 0$ for any group H. Then by [2, Theorem C] we have $\check{H}^{n+1}(X_A\times Y, X_A\times F:G)\supset \check{H}^1(Y, F:\check{H}^n(X_A:G))\neq 0$. This implies $D(X_A\times Y:G)\geqq n+1$.

For every positive integer n, Anderson and Keisler [1] constructed a separable metric space X_n such that $\dim X_n=\dim X_n^\omega=n-1$, where X_n^ω is the denumerable product of X_n with itself. Hence we can not omit the condition of the compactness of X in Theorem 4.

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