Papers Contributed

15. *On the Mutual Reduction of Algebraic Equations.*

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Two algebraic equations $F(x)=\overset{\mu}{\Pi}(x-\alpha_\mu)=0$ and $G(y)=\overset{\nu}{\Pi}(y-\beta_\nu)=0$ of degrees m and n respectively, irreducible in the rationality-domain R, being given, let a polynomial $\varphi(x, y)$ with rational coefficients be so chosen, that the mn values $\gamma_{\mu\nu}=\varphi(\alpha_\mu, \beta_\nu)$ are different from each other. These values are the roots of an equation $H(z)=0$ of degree mn in R and $R(\gamma_{\mu\nu})=R(\alpha_\mu, \beta_\nu)$.

Then we have the following very simple and interesting theorem, which seems to have remained unnoticed.

If $H(z)$ breaks up in e factors $h_i(z)$, irreducible in R, and of degree l_i $(i=1, 2,\cdots\cdots, e)$, and if $f_i(x, \beta)$ is the greatest common divisor of $F(x)$ and $h_i[x, \beta]$, and $g_i(y, \alpha)$ of $G(y)$ and $h_i[\alpha, y]$, then

$$F(x)=f_1(x, \beta)f_2(x, \beta)\cdots\cdots f_e(x, \beta),$$

$$G(y)=g_1(y, \alpha)g_2(y, \alpha)\cdots\cdots g_e(y, \alpha)$$

give the decomposition into irreducible factors of $F(x)$ and $G(y)$ in $R(\beta)$ and $R(\alpha)$ respectively.—$h_i[x, y]$ stands for $h_i\{\varphi(x, y)\}$, α or β for any root of $F(x)=0$ or of $G(y)=0$. If $f_i(x, \beta)$ and $g_i(y, \alpha)$ are of degrees m_i and n_i respectively, then it is known that

$$l_i=m_i n=mn_i, \qquad \frac{m}{n}=\frac{m_i}{n_i}\ (i=1, 2,\cdots\cdots, e).$$

Proof is almost redundant. $\gamma_{\mu\nu}=\varphi(\alpha_\mu\beta_\nu)$ denoting a root of $h_i(z)=0$, $h_i[x, \beta_\nu]$ has with $F(x)$ the greatest common divisor of a degree, say $m'>0$, which, on account of the irreducibility of $G(y)=0$ in R, must be independent of ν, so that the greatest common divisor is $f_i(x, \beta_\nu)$ and $m'=m_i$. The total number of the common roots of $F(x)=0$ and $h_i[x, \beta_\nu]=0$, $\nu=1, 2,\cdots\cdots, n$, being l_i, we have $l_i=m_i n$. If now $h_i[\alpha, \beta]=0$

and $\gamma=\varphi(\alpha\beta)$, so that $R(\gamma)=R(\alpha, \beta)$, then it follows from $l_i=m_i n$, that γ, as well as α, must satisfy an equation of degree m_i, which is irreducible in $R(\beta)$, so that $f_i(x, \beta)=0$, being just of degree m_i, is necessarily irreducible in $R(\beta)$; similarly for $g(y, \alpha)$, q.e.d.

The relations $h_i[\alpha_\mu\beta_\nu]=0$, $f_i(\alpha_\mu\beta_\nu)=0$ and $g_i(\beta_\nu\alpha_\mu)=0$ subsisting at the same time, $f_i(x, \beta)$ can also be characterized as the greatest common divisor of $F(x)$ and $g_i(\beta, x)$, as $g_i(y, \alpha)$ of $G(y)$ and $f_i(\alpha, y)$, as has been shown by A. LOEWY.[1] But considering, as he does, only the mutual reduction of $F(x)$ and $G(y)$, the relation with the defining equation $H(z)=0$ of the corpus $R(\alpha, \beta)$ has been left out of consideration, which gap to fill was the object of the present Note.

1) A. LOEWY, Über die Reduktion algebraischer Gleichungen durch Adjunktion insbesondere reeller Radikale, Math. Zeitschr., 15 (1922), 261–273, s. in particular p. 266.