123. *On Zero Points of a Meromorphic Function of Finite Order.*

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The second theorem proved by Mr. TSUJI in his paper: "On the zero points of a bounded analytic function "[1], concerning the roots of a transcendental integral function of finite order, can be extended to the meromorphic function of finite order.

Let $f(z)$ be a meromorphic function of finite order[2], regular at $z=0$.

Let $\lambda_1(x), \lambda_2(x), \cdots$ denote the roots of $f(z)=x$ in ascending order of absolute values,

$$|\lambda_\nu(x)| = r_\nu(x), \quad (\nu=1, 2, \cdots)$$

and $n(r; x)$ the number of the roots of $f(z)=x$ in $|z|<r$ and especially

$$\lambda_\nu(\infty) = \lambda_\nu, \quad r_\nu(\infty) = r_\nu, \quad (\nu=1, 2, \cdots) \quad \text{and} \quad n(r; \infty) = n(r).$$

Then by similar method as in my paper "On the power series etc.", Japanese Journal of Math., 2 (1925), 88, we can prove the following theorem:

Theorem I. *Consider a set of functions $\{f(z)\}$, which have the following properties:*

(i) *$f(z)$ is meromorphic in $|z| \leqq r$,*

(ii) *$f(0)=1$,*

(iii) *$f(z)$ has m roots and $n(r)$ poles in $|z|<r$.*

Then among such functions the unique one corresponding to the minimum of the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log} |f(r e^{i\theta})| \, d\theta$$

1) These Proceedings, 2 (1926), 248.

2) The definition of order here used is due to R. NEVANLINNA. See R. NEVANLINNA, Zur Theorie der meromorphen Funktionen, Acta Math., 46 (1925), 23, or VALIRON, Fonctions entières et fonctions méromorphes d'une variable, Paris (1925), 37–38.

is

$$f(z)=r^{n-n(r)}\frac{\lambda_1\lambda_2\cdots\lambda_{n(r)}}{\lambda_1(0)\,\lambda_2(0)\cdots\lambda_m(0)}\prod_{\nu=1}^{m}\frac{r(\lambda_\nu(0)-z)}{r^2-\overline{\lambda_\nu(0)}z}\prod_{\nu=1}^{n(r)}\frac{r^2-\bar{\lambda}_\nu z}{r(\lambda_\nu-z)}.$$

Conversely, *if we replace the condition* (ii) *by*

$$\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\,|f(r\,e^{i\theta})|\,d\theta\leqq M,$$

we have

$$f(0)\leqq e^M\frac{r_1(0)\,r_2(0)\cdots r_m(0)}{r^m}\cdot\frac{r^{n(r)}}{r_1r_2\cdots r_{n(r)}},\quad(m\leqq n(r\,;0)),$$

$f(z)$ being supposed to be regular and not equal to zero at $z=0$.

The last inequality may be put in the form

$$\log\frac{|f(0)|\,r^m}{r_1(0)\,r_2(0)\cdots r_m(0)}\leqq\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\,|f(r\,e^{i\theta})|\,d\theta+\log\frac{r^{n(r)}}{r_1r_2\cdots r_{n(r)}}.$$

Since $f(z)$ is of order q, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\,|f(r\,e^{i\theta})|\,d\theta+\log\frac{r^{n(r)}}{r_1r_2\cdots r_{n(r)}}<r^{q'},\quad(q'>q,\ r\geqq R),$$

so that we get

$$r_1(0)\,r_2(0)\cdots r_m(0)>|f(0)|\,r^m\,e^{-r^{q'}},\quad(r\geqq R,\ m\leqq n(r\,;0)),$$

whence we have

$$r_m(0)>|f(0)|^{\frac{1}{m}}\,r\,e^{-\frac{r^{q'}}{m}},\quad(r\geqq R,\ m\leqq n(r\,;0)).$$

It can easily be seen that the last inequality holds for every value of $r\geqq R$, and hence m may be considered to be independent of $r\geqq R$.

Now the value of r which makes $r\,e^{-\frac{r^{q'}}{m}}$ maximum is easily found to be

$$r=\left(\frac{m}{q'}\right)^{\frac{1}{q'}},$$

so that we have

$$r_m(0)>|f(0)|^{\frac{1}{m}}\cdot\left(\frac{m}{eq'}\right)^{\frac{1}{q'}},\quad(m\geqq q'\,R^{q'}=m_0).$$

Similarly we can prove that

$$r_m(x)>|f(0)-x|^{\frac{1}{m}}\cdot\left(\frac{m}{eq'}\right)^{\frac{1}{q'}},\quad(m\geqq m_x,\ f(0)\neq x).$$

Hence

Theorem 2. *If $f(z)$ is a meromorphic function of order q and $\lambda_1(x)$, $\lambda_2(x)$,$\cdots$ are the roots of $f(z)=x$ in ascending order of absolute values, then*

$$|\lambda_m(x)| > |f(0)-x|^{\frac{1}{m}}\left(\frac{m}{eq'}\right)^{\frac{1}{q'}}, \quad (q'>q,\ m \geqq m_x),$$

in which $f(z)$ is supposed to be regular and not equal to x at $z=0$.

From this we can prove that *the series*

$$\sum_{n=1}^{\infty}\frac{1}{|\lambda_n(x)|^{q+\epsilon}}, \quad (\varepsilon>0)$$

is convergent for an arbitrary value of x except $x=f(0)$.