122. *Note on Mr. Tsuji's Theorem.*

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In these Proceedings, **2** (1926), 245, Mr. TSUJI proved an interesting theorem concerning the zero points of a bounded analytic function. Analogous theorems can be established for certain classes of non-bounded functions by similar method.

1. First let $f(z)$ be regular and analytic for $|z|<1$, and suppose that

$$f(0)=1, \quad \text{and} \quad \left\{\frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta\right\}^{\frac{1}{p}} \leqq M_p, \ (p\text{: real}).$$

We call such a function $f(z)$ a function of the class M_p.

If we put

$$r_n^{(p)}=\frac{1}{\sqrt[n]{M_p}},$$

we can prove that

(i) *Every function of the class M_p has at most $n-1$ roots in the circle $|z|<r_n^{(p)}$.*

(ii) *Among the functions of the class M_p there exists a function which has just n roots in the circle $|z|\leqq r_n^{(p)}$. This function must be of the form*

$$f(z)=\frac{1}{a_1a_2\cdots a_n}\cdot\frac{a_1-z}{1-\bar{a}_1z}\cdot\frac{a_2-z}{1-\bar{a}_2z}\cdots\frac{a_n-z}{1-\bar{a}_nz},$$

where

$$|a_\nu|=\frac{1}{\sqrt[n]{M_p}}, \quad (\nu=1,2,\cdots,n).$$

These properties can be proved if we use the inequality[1)]

$$|f(z)|\leqq M_p\frac{1}{\{1-|z|^2\}^{\frac{1}{p}}}\prod_{\nu=1}^{n}\left|\frac{a_\nu-z}{1-\bar{a}_\nu z}\right|$$

instead of JENSEN's in TSUJI's paper, where a_ν $(\nu=1,2,\dots,n)$ are the roots of $f(z)$ in $|z|<1$ in ascending order of absolute values.

1) S. TAKENAKA, On the power series whose values are given at given points, Japanese Journal of Mathematics, **2** (1925), 81.

In particular, if we make $p\to\infty$, we have TSUJI's theorem.

2. Next let $f(z)$ be regular and analytic for $|z|<1$, and suppose that

$$f(0)=1 \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log}|f(e^{i\theta})|\,d\theta \leqq \log M_0 \text{ [1]},$$

We call such a function $f(z)$ a function of the class M_0.

If a_ν $(\nu=1,2,\cdots,n)$ are the roots of $f(z)=0$ in ascending order of absolute values, we have

$$1=f(0)\leqq M_0\prod_{\nu=1}^{n}|a_\nu| \text{ [2]}.$$

Then if we put

$$r_n=\frac{1}{\sqrt[n]{M_0}},$$

we can prove that

(i) *Every function of the class M_0 has at most $n-1$ roots in the circle $|z|<r_n$.*

(ii) *Among such functions of the class M_0, there exists a function which has just n roots in the circle $|z|\leqq r_n$. This function must be of the form*

$$f(z)=\frac{1}{a_1a_2\cdots a_n}\cdot\frac{a_1-z}{1-\bar{a}_1z}\cdot\frac{a_2-z}{1-\bar{a}_2z}\cdots\frac{a_n-z}{1-\bar{a}_nz},$$

where

$$|a_\nu|=\frac{1}{\sqrt[n]{M_0}},\quad (\nu=1,2,\cdots,n)$$

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1) $\overset{+}{\log}\, a$ stands for $\log a$, if $a>1$, and 0, if $a\leqq 0$.

2) S. TAKENAKA, loc. cit., 90.