89. *Fundamental Forms in the Projective Differential Geometry of m-parametric Families of Hypersurfaces of the Second Order in the n-Dimensional Space.*

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1. As I have already reported in another place, it seems to be very natural, to consider hypersurfaces of the second order as space-elements, when we intend to establish the projective differential geometry in the n-dimensional space.[1] Therefore the first problem to deal with is the projective theory of m-parametric families of hypersurfaces of the second order. In this paper I shall determine the fundamental forms in the projective differential geometry of m-parametric families of hypersurfaces of the second order and then add their geometrical meanings.

2. *Notations.* In homogeneous point-coordinates x_λ ($\lambda=0, 1, 2, \ldots \ldots, n$) an m-parametric family of hypersurfaces of the second order is represented by the equation

$$\sum_{\lambda,\mu} \overline{a_{\lambda\mu}}\, x_\lambda x_\mu = 0,$$

where
$$\overline{a_{\lambda\mu}} = \overline{a_{\lambda\mu}}(u^1, u^2, \ldots\ldots, u^n).$$

Then let us take

$$a_{\lambda\mu} = \left\{(n+1)\Delta\right\}^{-\frac{1}{n+1}} \overline{a_{\lambda\mu}}$$

as its normalized coordinates, where we represent the determinant $|\overline{a_{\lambda\mu}}|$ by Δ. Now we denote briefly a system of numbers $a_{\lambda\mu}$ by $\mathfrak{a}$, fol-

1) See my previous papers: On the projective differential geometry of plane curves and one-parameter families of conics, these Proceedings, **2**, 307-309, 1926; and Projective differential geometrical properties of the one-parameter families of point-pairs in the one-dimensional space, these Proceedings, **3**, 6-8, 1927. See also my papers: Über die projektive Differentialgeometrie I, II, III, Tôhoku Math. Journal, **28**, 1927.

lowing the vector-notation, and a system of numbers $n\,!\,A_{\lambda\mu}$ by $\mathfrak{A}$, where $A_{\lambda\mu}$ is the algebraic complement of $a_{\lambda\mu}$ in the determinant $|a_{\lambda\mu}|$. Put

$$(\mathfrak{a}_{i_0},\ \mathfrak{a}_{i_1},\ \mathfrak{a}_{i_2},\dots\dots,\ \mathfrak{a}_{i_n})=\sum\begin{vmatrix} a_{00}^{(i_0)} & a_{01}^{(i_1)} & \dots\dots\dots & a_{0n}^{(i_n)} \\ a_{10}^{(i_0)} & a_{11}^{(i_1)} & \dots\dots\dots & a_{1n}^{(i_n)} \\ \dots & \dots & \dots & \dots \\ a_{n0}^{(i_0)} & a_{n1}^{(i_1)} & \dots\dots\dots & a_{nn}^{(i_n)} \end{vmatrix},$$

where $\sum$ is extended over all permutations of $(i_0, i_1,\dots\dots, i_n)$. Moreover, we define the scalar product of two vectors $\mathfrak{a}$ and $\mathfrak{B}$ as follows:

$$\mathfrak{a}\mathfrak{B}=\sum_{\lambda,\mu} n\,!\,a_{\lambda\mu}\,B_{\lambda\mu}\,,$$

then we have

$$\mathfrak{a}\mathfrak{A}=(\mathfrak{a},\ \mathfrak{a},\dots\dots,\ \mathfrak{a})=1.$$

3. *Fundamental forms.* Let us consider

$$G_2\equiv g_{ik}du^i du^k=n(\mathfrak{a}_i,\ \mathfrak{a}_k,\ \mathfrak{a}\dots\dots,\ \mathfrak{a})du^i du^k\,,$$

$$A_3\equiv A_{ijk}du^i du^j du^k=(n-1)(\mathfrak{a}_i,\ \mathfrak{a}_j,\ \mathfrak{a}_k,\ \mathfrak{a},\dots\dots,\ \mathfrak{a})\,du^i du^j du^k.$$ [1]

These differential forms are evidently invariant under the group of unimodular projective transformations and under the change of parameters. Let us consider such vectors $\mathfrak{x}_\alpha$, $\mathfrak{X}^\beta$, that

$$\mathfrak{x}_\alpha\mathfrak{A}=\mathfrak{x}_\alpha\mathfrak{A}_i=\mathfrak{a}\mathfrak{X}^\beta=\mathfrak{a}_k\mathfrak{X}^\beta=0,$$

$$\mathfrak{x}_\alpha\mathfrak{X}^\beta=\delta_{\alpha\beta},$$ [2]

$$\left(\alpha=1,2,\dots\dots,\frac{(n+1)(n+2)}{2}-m-1\right),$$

and further consider g_{ik} as the fundamental tensor and introduce the covariant differentiation. Then the covariant derivatives of $\mathfrak{a}_i$ and $\mathfrak{A}_k$ can be represented as follows:

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1) Here $\mathfrak{a}_i=\dfrac{\partial\mathfrak{a}}{\partial u^i}$ and the repeated indices $i, j, k,\dots\dots$, one upper and another lower, are summed over $1, 2, \dots, m$ and $\alpha, \beta, \gamma,\dots$ over $1, 2, \dots, \dfrac{(n+1)(n+2)}{2}-m-1$.

2) $\delta_{\alpha\beta}=1$ for $\alpha=\beta$ and $\delta_{\alpha\beta}=0$ for $\alpha\neq\beta$.

$$(1) \qquad \mathfrak{a}_{ij} = -g_{ij}\mathfrak{a} - \frac{1}{2}A_{ijk}g^{kl}\mathfrak{a}_l + B_{ij}^{\cdot\cdot\alpha}\mathfrak{x}_\alpha ,$$

$$\mathfrak{A}_{kj} = -g_{kj}\mathfrak{A} + \frac{1}{2}A_{ijk}\,g^{jl}\,\mathfrak{A}_l + \overline{B}_{ik\beta}\,\mathfrak{X}^\beta .$$

Moreover we get

$$(2) \qquad \mathfrak{x}_{\alpha,i} = -\overline{B}_{il\alpha}\,g^{lk}\mathfrak{a}_k + p_{i\alpha}^{\cdot\cdot\gamma}\mathfrak{x}_\gamma ,$$

$$\mathfrak{X}_k^\beta = -B_{kl}\,g^{li}\,\mathfrak{A}_i - p_{k\gamma}^{\cdot\cdot\beta}\mathfrak{X}^\gamma ,$$

where

$$\mathfrak{a}_{ij}\mathfrak{X}^\beta = B_{ij}^{\cdot\cdot\beta} , \qquad \mathfrak{A}_{ij}\mathfrak{x}_\alpha = \overline{B}_{ij\alpha} ,$$

$$\mathfrak{x}_\alpha\,\mathfrak{X}^\beta_{\cdot,k} = -\mathfrak{x}_{\alpha,k}\mathfrak{X}^\beta = p_{k\alpha}^{\cdot\cdot\beta} .$$

$p_{k\alpha}^{\cdot\cdot\beta}$ can be determined by the quantities g_{ik}, A_{ijk}, $B_{kl}^{\cdot\cdot\alpha}$, $\overline{B}_{kl\beta}$.

From (1) new differential forms

$$B_2^\alpha = B_{kl}^{\cdot\cdot\alpha}du^k du^l ,$$

$$\overline{B}_{2\beta} = \overline{B}_{kl\beta}\,du^k du^l$$

appear, besides G_2 and A_3. These forms are apparently invariant under the group of unimodular projective transformations and under the change of parameters, which we adopt also as the fundamental forms.

4. *Gauss-Codazzi relations (conditions of integrability).* From (1) and (2) we can easily derive the following relations :

$$(3) \qquad \mathfrak{a}_{ijm} = \frac{1}{2}A_{ijm}\mathfrak{a} - g_{ij}\mathfrak{a}_m + \left(\frac{1}{4}A_{ij}^{\cdot\cdot k}A_{km}^{\cdot\cdot l} - \frac{1}{2}A_{ij}^{\cdot\cdot l}{}_{,\dot m} - B_{ij}^{\cdot\cdot\alpha}\overline{B}_{m\cdot\alpha}^{\cdot l\cdot}\right)\mathfrak{a}_l$$

$$+\left(B_{ij}^{\cdot\cdot\alpha}{}_{,\dot m} + B_{ij}^{\cdot\cdot\beta}p_{m\beta}^{\cdot\cdot\alpha} - \frac{1}{2}A_{ij}^{\cdot\cdot k}B_{km}^{\cdot\cdot\alpha}\right)\mathfrak{x}_\alpha ,$$

$$\mathfrak{A}_{ijm} = \frac{1}{2}A_{ijm}\mathfrak{A} - g_{ij}\,\mathfrak{A}_m + \left(\frac{1}{4}A_{ij}^{\cdot\cdot k}A_{km}^{\cdot\cdot l} + \frac{1}{2}A_{ij}^{\cdot\cdot l}{}_{,\dot m} - \overline{B}_{ij\alpha}B_m^{\cdot l\alpha}\right)\mathfrak{A}_l$$

$$+\left(\overline{B}_{ij\alpha,m} - \overline{B}_{ij\beta}\,p_{m\alpha}^{\cdot\cdot\beta} + \frac{1}{2}A_{ij}^{\cdot\cdot k}\overline{B}_{km\alpha}\right)\mathfrak{X}^\alpha ,$$

where

$$A_{ij}^{\cdot\cdot l} = g^{lk}A_{ijk} , \qquad B_m^{\cdot k} = g^{kl}B_{ml} , \quad \text{etc.}$$

From (3) we get by simple calculation

$$(4) \qquad \mathfrak{a}_{i[j\dot m]} = \left(\frac{1}{4}A_{i[j}^{\cdot\cdot k}A_{\dot m]k}^{\cdot\cdot l} - \frac{1}{2}A_{i[j}^{\cdot\cdot l}{}_{,\dot m]} - B_{i[j}^{\cdot\cdot\alpha}\,\overline{B}_{\dot m]\cdot\alpha}^{\cdot l\cdot}\right)\mathfrak{a}_l$$

$$+\left(B_{i(j}{}^{\alpha}{}_{;\,m)}+B_{i(j}{}^{\beta}p_{m)\beta}{}^{\alpha}-\frac{1}{2}A_{i(j}{}^{k}B_{m)k}{}^{\alpha}\right)\mathfrak{x}_{\alpha}\,,^{1)}$$

$$\mathfrak{A}_{i(jm)}=\left(\frac{1}{4}A_{i(j}{}^{k}A_{m)k}{}^{l}+\frac{1}{2}A_{i(j}{}^{l}{}_{,\,m)}-B_{(m}{}^{l\alpha}\overline{B}_{j)i\alpha}\right)\mathfrak{A}_{l}$$

$$+\left(\overline{B}_{i(j\,|\,\alpha\,|,\,m)}-\overline{B}_{i(j\,|\,\beta\,|}\,p_{m)\alpha}{}^{\beta}+\frac{1}{2}A_{i(j}{}^{k}\overline{B}_{m)k\alpha}\right)\mathfrak{X}^{\alpha}.$$

But by the theorem of Ricci we know that the following relations hold good:

$$\text{(5)}\qquad \mathfrak{a}_{i(jm)}=K_{jmi}{}^{l}\mathfrak{a}_{l}\,,\qquad \mathfrak{A}_{i(jm)}=K_{jmi}{}^{l}\,\mathfrak{A}_{l}\,,$$

where $K_{jmi}{}^{l}$ denotes Riemann's curvature-tensor. Therefore we must have

$$\text{(6)}\qquad \frac{1}{4}A_{i(j}{}^{k}A_{m)k}{}^{l}-\frac{1}{2}A_{i(j}{}^{l}{}_{,\,m)}-B_{i(j}{}^{\alpha}\overline{B}_{m)}{}^{l}{}_{\alpha}=K_{jmi}{}^{l}\,,$$

$$\text{(7)}\qquad \frac{1}{4}A_{i(j}{}^{k}A_{m)k}{}^{l}+\frac{1}{2}A_{i(j}{}^{l}{}_{,\,m)}-B_{(m}{}^{l\alpha}\,\overline{B}_{j)i\alpha}=K_{jmi}{}^{l}\,,$$

$$\text{(8)}\qquad B_{i(j}{}^{\alpha}{}_{,\,m)}+B_{i(j}{}^{\beta}p_{m)\alpha}{}^{\beta}-\frac{1}{2}A_{i(j}{}^{k}B_{m)k}{}^{\alpha}=0,$$

$$\overline{B}_{i(j\,|\,\alpha\,|,\,m)}-\overline{B}_{i(j\,|\,\beta\,|}\,p_{m)\alpha}{}^{\beta}+\frac{1}{2}A_{i(j}{}^{k}\overline{B}_{m)k\alpha}=0.$$

From (6) and (7), which are essentially identical with each other,

$$\text{(9)}\qquad \frac{1}{4}A_{i(j}{}^{k}A_{m)k}{}^{l}-\frac{1}{2}\left(B_{i(j}{}^{\alpha}\overline{B}_{m)}{}^{l}{}_{\alpha}+B_{m}{}^{l\alpha}\overline{B}_{j)i\alpha}\right)=K_{jmi}{}^{l}\,,$$

$$A_{i(j}{}^{l}{}_{,\,m)}=B_{(m}{}^{l\alpha}\overline{B}_{j)i\alpha}-B_{i(j}{}^{\alpha}\overline{B}_{m)}{}^{l}{}_{\alpha}.$$

These relations (8) and (9) are the equations, which correspond to the so-called Gauss-Codazzi equations, i.e. the conditions of integrability in our case.

5. *The fundamental theorem.* We can now prove the fundamental theorem:

The famliy of hypersurfaces of the second order is uniquely determined, except for the projective transformations, by the forms G_2, A_3, $B_2{}^{\alpha}$, $\overline{B}_{2\beta}$ and $p_{i\alpha}{}^{\beta}$, *among which the relations* (8) *and* (9) *hold.*

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1) We introduce the following notation after Schouten: $\mathfrak{K}_{(ij)}=\mathfrak{K}_{ij}-\mathfrak{K}_{ji}$ for ex. $\mathfrak{A}_{i(jm)}=\mathfrak{A}_{ijm}-\mathfrak{A}_{imj}$, $A_{i(j}{}^{k}A_{m)k}{}^{l}=A_{ij}{}^{k}A_{mk}{}^{l}-A_{im}{}^{k}A_{jk}{}^{l}$, $\overline{B}_{i(j\,|\,\alpha\,|,\,m)}=\overline{B}_{ij\alpha,m}-\overline{B}_{im\alpha,j}$, etc.

6. *Other relations.* From (2) we have

$$\mathfrak{x}_{\alpha k}\mathfrak{X}_{l}^{\beta}=\overline{B}_{kj\alpha}B_{l}^{\cdot j\beta}-p_{k\dot{\alpha}}{}^{\gamma}p_{l\dot{\gamma}}{}^{\beta}\,,$$

and

$$\mathfrak{x}_{\alpha k}\mathfrak{X}_{l}^{\beta}+\mathfrak{x}_{\alpha kl}\,\mathfrak{X}^{\beta}=p_{k\dot{\alpha}}{}^{\beta}{}_{,\,\dot{m}}\,,$$

i. e.

$$\mathfrak{x}_{\alpha(k}\,\mathfrak{X}^{\beta}{}_{,\,l)}=p_{(\dot{k}\,|\,\dot{\alpha}\,|}{}^{\beta}{}_{,\,m)}=\overline{B}_{i(k\,|\,\alpha\,|}B_{l)}^{\cdot i\beta}\,, \tag{10}$$

and also from (1)

$$\mathfrak{a}_{(k}^{\cdot i}\,\mathfrak{A}_{l)i}=B_{(k}^{\cdot i\alpha}\,\overline{B}_{l)i\alpha}=p_{(\dot{k}\,|\,\dot{\alpha}\,|}{}^{\alpha}{}_{,\,m)}\,. \tag{11}$$

7. *Geometrical meaning of the fundamental forms.* Now we consider the geometrical meaning of the fundamental forms. First the ∞^{m-1} directions defined by $G_2=0$ are such that the hypersurfaces of the second order $d\mathfrak{a}$, which belong to the sheaf determined by $\mathfrak{a}$ and the consecutive hypersurfaces of the second order $\mathfrak{a}+d\mathfrak{a}$ in that directions and which have apolarity of the first order[1] to $\mathfrak{a}$, have also apolarity of the second order to $\mathfrak{a}$. The ∞^{m-1} directions defined by $A_3=0$ are such that the hypersurfaces $d\mathfrak{a}$, above mentioned, have apolarity of the third order to $\mathfrak{a}$. $B_2^{\alpha}=0$ defines the ∞^{m-1} directions such that the hypersurfaces $d\mathfrak{a}$ have apolarity of the first order to $d\mathfrak{X}^{\alpha}$, and similarly for $\overline{B}_{2\beta}=0$.

In the special case $n=2$, $G_2=0$ defines the directions in which $\mathfrak{a}$ is apolar to the conics $d\mathfrak{a}$, and $A_3=0$ the directions in which every $d\mathfrak{a}$ reduces to two straight lines.

8. *Projective principal hypersurfaces of the second order.* At every $\mathfrak{a}$ of the family we consider the hypersurfaces of the second order defined by

$$\mathfrak{p}=g^{ij}\mathfrak{a}_{ij}=-m\mathfrak{a}-\frac{1}{2}g^{ij}\mathrm{A}_{ij}^{\cdot\cdot l}\mathfrak{a}_{l}+g^{ij}B_{ij}^{\cdot\cdot\alpha}\mathfrak{x}_{\alpha}\,, \tag{12}$$

$$\mathfrak{P}=g^{ij}\mathfrak{A}_{ij}=-m\,\mathfrak{A}+\frac{1}{2}g^{ij}A_{ij}^{\cdot\cdot l}\mathfrak{A}_{l}+g^{ij}\overline{B}_{ij\beta}\,\mathfrak{X}^{\beta}\,,$$

which are invariant under the group of unimodular projective transformations as well as under the change of the parameters. So we call them the *principal direct and correlative hypersurfaces of the second order* at $\mathfrak{a}$ of the family.

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1) Apolarity of the p-th order to $\mathfrak{a}$ means that $(\mathfrak{b}, \mathfrak{b}, \ldots\ldots, \mathfrak{b}, \mathfrak{a}, \ldots\ldots, \mathfrak{a})=0$, in which the number of the $\mathfrak{b}$'s is p. Especially apolarity of the first order is the apolarity, in the usual sense, that is, the circumscribing of a self-polar $(n+1)$-polytope of $\mathfrak{a}$.