

88. *Analytic Proof of Blaschke's Theorem on the Curve of Constant Breadth with Minimum Area.*

By Matsusaburô FUJIWARA, M. I. A.

Mathematical Institute, Tohoku Imperial University, Sendai.

(Rec. May 31, 1927. Comm. June 12, 1927.)

That the Reuleaux triangle consisting of three circular arcs of radius a is a curve of constant breadth a with minimum area was geometrically proved by Prof. Blaschke in *Mathematische Annalen* 76, 1915. The aim of this note is to prove this theorem analytically.

Take a point on a given curve C of constant breadth a as the origin and a supporting line (Stützgerade) at this point as the initial line. Then the curve C may be represented by the polar-tangential equation of the form $p=p(\theta)$, where $p(0)=p'(0)=0$. As already shown by Prof. Kakeya,¹⁾ the curve of constant breadth a is characterized by the relations

$$\int_0^\pi \rho(\theta) \sin \theta \, d\theta = a, \quad \int_0^\pi \rho(\theta) \cos \theta \, d\theta = 0,$$

$$0 \leq \rho(\theta) \leq a, \quad \rho(\theta) + \rho(\theta + \pi) = a,$$

where $\rho(\theta)$ denotes the radius of curvature and satisfies

$$\rho(\theta) = p(\theta) + p''(\theta), \quad p(\theta) = \int_0^\theta \rho(\varphi) \sin(\theta - \varphi) \, d\varphi.$$

The area S of C being equal to $\frac{1}{2} \int_0^{2\pi} p(\theta) \rho(\theta) \, d\theta$, our problem may be formulated analytically as follows :

$$S = \frac{1}{2} \int_0^{2\pi} d\theta \int_0^\theta \rho(\theta) \rho(\varphi) \sin(\theta - \varphi) \, d\varphi = \text{Minimum},$$

1) Fujiwara-Kakeya, On some problems of maxima and minima for the curve of constant breadth and the in-revolvable curve of the equilateral triangle, *Tôhoku Math. Journ.*, 11 (1917).

under the conditions

$$\int_0^\pi \rho(\theta) \sin \theta \, d\theta = a, \quad \int_0^\pi \rho(\theta) \cos \theta \, d\theta = 0,$$

$$0 \leq \rho(\theta) \leq a, \quad \rho(\theta) + \rho(\theta + \pi) = a.$$

Such a problem lies beyond the scope of the elementary theory of maxima and minima in the infinitesimal calculus as well as the classical theory of calculus of variations. The main difficulty is due to the presence of inequality as auxiliary condition. There is, however, a powerful method due to Markoff,¹⁾ which is applicable to the problems of the following form :

$$\int_\alpha^\beta \rho(\theta) \varphi(\theta) \, d\theta = \text{Extremum},$$

under the conditions

$$\int_\alpha^\beta \rho(\theta) \varphi_k(\theta) \, d\theta = c_k, \quad (k=1, 2, \dots, n)$$

and

$$p \leq \rho(\theta) \leq q,$$

where $\varphi(\theta)$, $\varphi_k(\theta)$ are given functions.

Perceiving the importance of this method I applied it many years ago²⁾ to some extremum problems for ovals, but I could not solve the problem which is now in consideration. I remark now that our problem is reducible to Markoff's problem, where $\varphi(\theta)$ contains the unknown function $\rho(\theta)$.

The Reuleaux triangle is characterized by $\rho = \rho_0(\theta)$, defined as follows :

$$\begin{aligned} \rho_0(\theta) &= a && \text{for } 0 \leq \theta < \frac{\pi}{3}, \\ \rho_0(\theta) &= 0 && \text{for } \frac{\pi}{3} \leq \theta < \frac{2\pi}{3}, \\ \rho_0(\theta) &= a && \text{for } \frac{2\pi}{3} \leq \theta < \pi. \end{aligned}$$

1) Markoff, Recherches sur les valeurs extrêmes des intégrales et sur interpolation, Acta Mathematica, 28 (1904).

2) Fujiwara, Ueber die innen-umdrehbare Kurve eines Vieleckes, Science Reports of Tôhoku Imp. University, 8 (1919).

Let S_0 be its area. Then, observing that

$$\int_0^\pi \{\rho(\theta) - \rho_0(\theta)\} (A \cos\theta + B \sin\theta) d\theta = 0$$

holds good for any arbitrary constants A and B , we get after some elementary calculations

$$\begin{aligned} S - S_0 &= \int_0^{\frac{\pi}{3}} \{\rho(\theta) - \rho_0(\theta)\} (L(\theta) + a) d\theta \\ &+ \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \{\rho(\theta) - \rho_0(\theta)\} \left(L(\theta) + a \cos\left(\frac{\pi}{3} - \theta\right) \right) d\theta \\ &+ \int_{\frac{2\pi}{3}}^\pi \{\rho(\theta) - \rho_0(\theta)\} (L(\theta) + a(1 + \cos\theta)) d\theta, \end{aligned}$$

where $L(\theta) = \int_0^\pi \rho(\varphi) \sin(\theta - \varphi) d\varphi + A \cos\theta + B \sin\theta - a$.

$$\begin{aligned} \text{Since } \rho(\theta) - \rho_0(\theta) &\leq 0 \quad \text{for } 0 \leq \theta < \frac{\pi}{3}, \quad \frac{2\pi}{3} \leq \theta < \pi, \\ &\geq 0 \quad \text{for } \frac{\pi}{3} \leq \theta < \frac{2\pi}{3}, \end{aligned}$$

$S - S_0 \geq 0$, if we can determine A and B such that

$$L(\theta) + a \leq 0 \quad \text{for } 0 \leq \theta < \frac{\pi}{3},$$

$$L(\theta) + a \cos\left(\frac{\pi}{3} - \theta\right) \geq 0 \quad \text{for } \frac{\pi}{3} \leq \theta < \frac{2\pi}{3},$$

$$L(\theta) + a(1 + \cos\theta) \leq 0 \quad \text{for } \frac{2\pi}{3} \leq \theta < \pi,$$

which can be verified by some artifices.

Thus Blaschke's theorem is proved analytically.