11. *On the Series of Orthogonal Functions.*

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Let $c_1, c_2, \ldots\ldots$ be a sequence of real numbers and $\varphi_1(x), \varphi_2(x), \ldots\ldots$ a sequence of normalized orthogonal functions in the interval $(0, 1)$; then relating to the series

$$(1) \qquad c_1\varphi_1(x) + c_2\varphi_2(x) + \ldots\ldots$$

we have the following theorems :

(A) If the series $\sum(\log\nu)^2 . c_\nu^2$ is convergent, the series (1) converges almost everywhere in the interval $(0, 1)$[1].

(B) If the series $\sum(\log\log\nu)^2 . c_\nu^2$ is convergent, the series (1) is C_1-summable almost everywhere in the interval $(0, 1)$[2].

The author proved that (A) and (B) are deducible from the following theorem, due to Borgen and Kaczmarz[2].

(C) If the series $\sum(\log\log\nu)^2 . c_\nu^2$ is convergent, the sequence of the partial sums

$$(2) \qquad s(x, 2),\ s(x, 2^2),\ \ldots\ldots,\ s(x, 2^n),\ \ldots\ldots$$

converges almost everywhere in the interval $(0, 1)$, where

$$s(x, p) = c_1\varphi_1(x) + c_2\varphi_2(x) + \ldots\ldots + c_p\varphi_p(x).$$

Conversely, it can be shown that the theorem (C) is deducible from (A), and also from (B).

Thus, we know that each of the theorems (A) and (B) is deducible from the other ; i. e.

The theorems (A) *and* (B) *are equivalent.*

Corresponding to Lebesgue's constants in the theory of trigonometric series, we have for the system of normalized orthogonal functions $\varphi_1(x), \varphi_2(x), \ldots\ldots$, Lebesgue's functions

1) H. Rademacher, Math. Ann. **87** (1922) ; D. Menchoff, Fundamenta Mathematicae **4** (1923).

2) D. Menchoff, Comptes Rendus **180** (1925) ; S. Borgen, Math. Ann. **98** (1927) ; S. Kaczmarz, Math. Zeits. **26** (1927).

$$\rho_n(x) = \limsup \int_0^1 f(y) \sum_{\nu=1}^{n} \varphi_\nu(x)\, \varphi_\nu(y)\, dy \qquad (n = 1, 2, \ldots.),$$

for any $f(x)$ such that $|f(x)| \leqq 1,\ 0 \leqq x \leqq 1$.

Rademacher[1] has shown that

$$\rho_n(x) = \int_0^1 \left| \sum_{\nu=1}^{n} \varphi_\nu(x)\, \varphi_\nu(y) \right| dy = O\left((\log n)^{\frac{3}{2}+\varepsilon} \cdot n^{\frac{1}{2}} \right),$$

almost everywhere in (0, 1), and that

$$\rho_n = O\left(n^{\frac{1}{2}} \right),$$

when $\rho_1, \rho_2, \ldots\ldots$ are independent of x.

We can, however, prove the following

Theorem. *Except in a null-set, Lebesgue's functions* $\rho_\nu(x)$ $(n=1, 2, \ldots..)$ *of the normalized orthogonal functions* $\varphi_1(x), \varphi_2(x), \ldots.$ *satisfy the relation*

$$\rho_n(x) = o\left((\log n)^{\frac{1}{2}+\varepsilon} \cdot n^{\frac{1}{2}} \right), \qquad \varepsilon > 0,$$

in the interval of orthogonality ; and

$$\rho_n \leqq n^{\frac{1}{2}},$$

when the functions $\rho_n(x)$ $(n=1, 2, \ldots..)$ *are constants.*

As regards the order of the functions

$\varphi_n(x, y) = \varphi_1(x)\, \varphi_1(y) + \varphi_2(x)\, \varphi_2(y) + \ldots.. + \varphi_n(x)\, \varphi_n(y)$, $(n=1, 2, \ldots.)$,
Bossolasco[2] proved that $\varphi_n(x, y) = o(n)$ almost everywhere in the square Q: $0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1$. We can, however, *improve his result to*

$$\varphi_n(x, y) = o\left((\log n)^{\frac{3}{2}+\varepsilon} \cdot n^{\frac{1}{2}} \right), \qquad \varepsilon > 0,$$

almost everywhere in the square Q.

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1) H. Rademacher, loc. cit.

2) M. Bossolasco, Atti Accad. Torino, **62** (1927).