PAPERS COMMUNICATED

57. *On a Characteristic Property of the Sections of Some Power Series.*

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Theorem. Let

(1) $$f_{km}(x)=1+a_1x+a_2x^2+\ldots\ldots+a_{km}x^{km}$$

be the km-th section of a power series

(2) $$f(x)=1+a_1x+a_2x^2+\ldots\ldots+a_nx^n+\ldots\ldots$$

where k is a given positive integer.

If $f_{km}(x)=0$ has all their roots on $|x|=1$ for $m=1, 2,\ldots,$ then we have

(3) $$f(x)=\frac{1+a_1xe^{i\theta}+\ldots\ldots+a_{k-1}(xe^{i\theta})^{k-1}}{1-(xe^{i\theta})^k}$$

with the conditions that

(4) $$a_{k-i}=\bar{a}_i \quad \begin{pmatrix} i=1, 2,\ldots \dfrac{k}{2} & \text{when } k \text{ is even} \\ i=1, 2,\ldots \dfrac{k-1}{2} & \text{when } k \text{ is odd} \end{pmatrix}$$

and that

(5) $1+a_1x+\ldots\ldots+a_{k-1}x^{k-1}=0$ has all its roots in $|x|\geqq 1$.

And conversely.

Proof. By the hypothesis we have $|a_{km}|=1$, and we can put $a_k=1$. Further

(6) $$a_0\bar{a}_\nu=\bar{a}_{km}a_{km-\nu} \quad (\nu=1, 2,\ldots.km)$$

Hence

$$a_{km}=1 \;\; m=2, 3,\ldots.) \quad \text{and} \quad a_{k-i}=\bar{a}_i, \; (i=1, 2,\ldots.k-1).$$

Considering $f_{2k}(x)=0$, $f_{3k}(x)=0,\dots.$ successively, we have

$$f_{km}(x)=1+a_1x+\dots+a_{k-1}x^{k-1}+x^k+a_1x^{k+1}+\dots+a_{k-1}x^{2k-1}+x^{2k}+\dots$$
$$+a_1x^{(m-1)k+1}+\dots.+a_{k-1}x^{mk-1}+x^{mk}$$
$$=(1+a_1x+\dots.+a_{k-1}x^{k-1})\frac{1-x^{km}}{1-x^k}+x^{km}.$$

Hence

$$(7)\quad 1+a_1x+\dots+a_{k-1}x^{k-1}-x^{km+1}(a_1+a_2x+\dots+a_{k-1}x^{k-2}+x^{k-1})=0$$

has also all the roots on $|x|=1$. From (7)

$$(8)\quad x^{km+1}=\frac{1+a_1x+\dots+a_{k-1}x^{k-1}}{a_1+a_2x+\dots+a_{k-1}x^{k-2}+x^{k-1}}$$
$$=\frac{(1+\alpha_1x)(1+\alpha_2x)\dots(1+\alpha_{k-1}x)}{(\overline{\alpha_1}+x)(\overline{\alpha_2}+x)\dots(\overline{\alpha_{k-1}}+x)}.$$

If any one of the α's, say α_1, lie outside of the unit circle, then by Rouché's theorem (8) must have at least one root in the sufficiently small neighbourhood of $-\frac{1}{\alpha_1}$. If $|\alpha_1|, |\alpha_2|,\dots.|\alpha_{k-1}|\leqq 1$, then the right hand side of the equation (8) can not be equal to the left in absolute value, unless $|x|=1$. For $m\to\infty$ we have the theorem.

Remark.[1] Putting $P(x)=1+a_1x+\dots.+a_{k-1}x^{k-1}$, $(a_{k-i}=\overline{a_i})$, and remarking that $P(x)+x^k=f_k(x)=0$ has all its roots on $|x|=1$, we have

$$P(x)+x^k=x\overline{P}\left(\frac{1}{x}\right)+1.$$

Hence, if $P(x)=0$ has any root β such as $|\beta|=1$, then we must have

$$\beta^k=1$$

By the continuity of roots the condition (5) can be expressed by the following k inequalities

$$P(1)\geqq 0,\ P(\beta)\geqq 0,\dots.P(\beta^{k-1})\geqq 0,$$

where β is a primitive root of $\beta^k=1$.

1) For $k=1$ the above theorem was obtained by Jentzsch, Acta math. **41** (1918) 257.