37. On the Roots of the Characteristic Eqution of a Certain Matrix.

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The well-known theorem of Frobenius, that all the roots of the characteristic equation of a unitary matrix are of absolute value 1 is recently proved by Mr. H. Aramata[1] and Mr. R. Brauer[2] simply. I will here give another simple proof and a generalization of it.

Theorem. Let the transformation,

$$\begin{cases} X_1=a_{11}x_1+\dots+a_{1n}x_n\,, \\ \dots\dots\dots\dots\dots\dots \\ X_n=a_{n1}x_1+\dots+a_{nn}x_n\,, \end{cases} \qquad \begin{cases} \overline{X}_1=\overline{a}_{11}\overline{x}_1+\dots+\overline{a}_{1n}\overline{x}_n\,, \\ \dots\dots\dots\dots\dots\dots \\ \overline{X}_n=\overline{a}_{n1}\overline{x}_1+\dots+\overline{a}_{nn}\overline{x}_n\,, \end{cases}$$

$\overline{a}_{ik}$, $\overline{x}_i$ being conjugate complex of a_{ik} and x_i, make a function $F(x_1, \overline{x}_1, \dots x_n, \overline{x}_n)$ invariant, such that

$$(1) \qquad F(X_1, \overline{X}_1 \dots X_{n1} \overline{X}_n)=F(x_1, \overline{x}_1 \dots x_n, \overline{x}_n)\,,$$

where F satisfies the following conditions:

(i) $F(\lambda x_1, \overline{\lambda}\overline{x}_1 \dots \lambda x_n, \overline{\lambda}\overline{x}_n)=|\lambda|^k F(x_1, \overline{x}_1 \dots x_n, \overline{x}_n)$, *$k$ being a real number.*

(ii) $F(x_1, \overline{x}_1 \dots x_n, \overline{x}_n) \neq 0, \quad \neq \infty$ *for* $|x_1|+|x_2|+\dots+|x_n|>0$.

Then all the roots of the characteristic equation of the matrix $A=(a_{ik})$ are of absolute value 1.

When $F=x_1\overline{x}_1+\dots+x_n\overline{x}_n$, A becomes a unitary matrix.

Proof. Let λ be a root of the characteristic equation of A. Then the linear equations,

$$\begin{cases} \lambda x_1=a_{11}x_1+\dots+a_{1n}x_n\,, \\ \dots\dots\dots\dots\dots\dots \\ \lambda x_n=a_{n1}x_1+\dots+a_{nn}x_n\,, \end{cases}$$

has a solution $(x_1, x_2, \dots x_n)$ such that

1) H. Aramata, Über einen Satz für unitäre Matrizen, The Tôhoku Mathematical Journal **28** (1927), 281.

2) R. Brauer, Über einen Satz für unitäre Matrizen, The Tohoku Mathematical Journal, **30** (1928), 72.

$$(2) \qquad |x_1| + |x_2| + \ldots + |x_n| \neq 0 .$$

For such a solution, we have by (1)

$$F(\lambda x_1, \bar{\lambda} \bar{x}_1, \ldots \lambda x_n, \bar{\lambda} \bar{x}_n) = F(x_1, \bar{x}_1, \ldots x_n, \bar{x}_n)$$

and from (i)

$$F(\lambda x_1, \bar{\lambda} \bar{x}_1, \ldots \lambda x_n, \lambda \bar{x}_n) = |\lambda|^k F(x_1, \bar{x}_1, \ldots x_n, \bar{x}_n) .$$

Hence

$$|\lambda|^k F(x_1, \bar{x}_1 \ldots x_n, \bar{x}_n) = F(x_1, \bar{x}_1, \ldots x_n, \bar{x}_n)$$

and by (ii) and (2), we have

$$|\lambda|^k = 1 \quad \text{or} \quad |\lambda| = 1, \quad \text{q.e.d.}$$