PAPERS COMMUNICATED

17. *On the Expansion of an Integral Transcendental Function of the First Order in Generalized Taylor's Series.*

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(Comm. by M. FUJIWARA, M.I.A., March 12, 1932)

1. In my previous paper[1] I have proved the following theorem:

THEOREM A. *Let* $\{a_n\}$ *be a set of points such that*

$$\overline{\lim_{n=0}}|a_n|=L<\infty$$

Then any function $\phi(z)$, *regular and analytic for* $|z|<r$, *can be expanded in one and only one way into the series of the form*

$$\phi(z)=\sum_{n=0}^{\infty}c_nz^ne^{\bar{a}_nz} \tag{1. 1}$$

which converges absolutely and uniformly for $|z|\leqq r_0<\min\left(r,\ \frac{1}{eL}\right)$.

Let us define a sequence $\{p_n(z)\}$ of polynomials by

$$p_0(z)=1\,,\quad p_n(z)=\int_{a_0}^{z}\int_{a_1}^{t_1}\cdots\cdots\int_{a_{n-1}}^{t_{n-1}}dt_ndt_{n-1}\cdots\cdots dt_1\,,\quad (n\geqq 1) \tag{1. 2}$$

which satisfy the equalities:

$$p_n^{(\nu)}(a_\nu)=\begin{cases}0 & \text{for} \quad \nu\neq n\,,\\ 1 & \text{for} \quad \nu=n\,,\end{cases} \tag{1. 3}$$

and put

$$p_n(z)=\sum_{\nu=0}^{n}\frac{k_\nu^{(n)}}{\nu\,!}z^\nu\,,\qquad (n=0,\ 1,\ 2,\ \ldots\ldots)\,,$$

and define a sequence $\{\pi_n(z)\}$ of polynomials by

$$\pi_n(z)=\sum_{\nu=0}^{n}k_\nu^{(n)}z^\nu\,,\qquad (n=0,\ 1,\ 2,\ \ldots\ldots)\,.$$

Then it can easily be shown that

$$\begin{cases}p_n(z)=\dfrac{1}{2\pi}\displaystyle\int_{|\zeta|=1}\pi_n(\zeta)e^{z\bar{\zeta}}\,|d\zeta|\,,\\[2ex] p_n^{(\nu)}(a_\nu)=\dfrac{1}{2\pi}\displaystyle\int_{|\zeta|=1}\pi_n(\zeta)\bar{\zeta}^\nu e^{a_\nu\bar{\zeta}}\,|d\zeta|\,,\qquad (n,\ \nu=0,\ 1,\ \ldots\ldots)\,,\end{cases} \tag{1. 4}$$

1) S. Takenaka: On the distribution of zero points of the derivatives of an integral transcendental function of order $\rho\leqq 1$, Proc. **7** (1931), 134.

so that, from the equalities (1. 3),

$$\frac{1}{2\pi}\int_{|z|=1}\pi_n(z)\bar{z}^{\nu}e^{\alpha_\nu\bar{z}}|dz|=\begin{cases}0 & \text{for } \nu\neq n,\\ 1 & \text{for } \nu=n,\end{cases}$$

from which we see that the sequence of polynomials

$$\pi_n(z), \qquad (n=0, 1, 2, \ldots\ldots)$$

and the sequence of functions

$$z^n e^{\bar{\alpha}_n z}, \qquad (n=0, 1, 2, \ldots\ldots)$$

are each other biorthogonal on $|z|=1$.

Now, in Theorem A, let us put $r=1+\varepsilon$ (ε being an arbitrary small positive constant) and $L<e^{-1}$. Then the series of the right hand side of (1. 1) converges absolutely and uniformly for $|z|\leqq 1$.

Therefore multiplying the both sides by $\frac{1}{2\pi}\pi_n(z)$ and integrating term by term, we get

$$c_n=\frac{1}{2\pi}\int_{|z|=1}\phi(z)\overline{\pi_n(z)}\,|dz|, \qquad (n=0, 1, 2, \ldots\ldots)$$

from which we can state the theorem:

THEOREM I. *Let $\{\alpha_n\}$ be a set of points such that*

$$\overline{\lim_{n=\infty}}|\alpha_n|=L<\frac{1}{e}$$

Then any function $\phi(z)$, regular and analytic for $|z|<r$, $(r>1)$ can be expanded in one and only one way into the series of the form

$$(1.\ 5)\qquad \phi(z)=\sum_{n=0}^{\infty}c_n z^n e^{\bar{\alpha}_n z}, \qquad c_n=\frac{1}{2\pi}\int_{|z|=1}\phi(z)\overline{\pi_n(z)}\,|dz|,$$

which converges absolutely and uniformly for $|z|\leqq r_0<\min\left(r, \frac{1}{eL}\right)$, where $\{p_n(z)\}$ and $\{\pi_n(z)\}$ are defined by (1. 2) *and* (1. 4) *respectively.*

2. In (1. 5), if we put

$$\phi(z)=e^{\bar{x}z}, \qquad (x \text{ being any complex number}),$$

we have (from (1. 4))

$$c_n=\frac{1}{2\pi}\int_{|z|=1}e^{\bar{x}z}\overline{\pi_n(z)}\,|dz|=\overline{p_n(x)}, \qquad (n=0, 1, 2, \ldots\ldots).$$

Whence we get

$$e^{\bar{x}z}=\sum_{n=0}^{\infty}\overline{p_n(x)}z^n e^{\bar{\alpha}_n z},$$

or

$$e^{x\bar{z}}=\sum_{n=\infty}^{\infty}p_n(x)\bar{z}^n e^{\alpha_n\bar{z}},$$

which converges absolutely and uniformly for $|z| \leqq r_0 < \frac{1}{eL}$.

For the convenience sake let us write $\{\sigma_n\}$ in the place of $\{a_n\}$ under the condition that

$$\overline{\lim_{n=\infty}} |\sigma_n| = l < \frac{1}{e}.$$

Then we have

$$(2.\ 1)\quad e^{x\bar{z}} = \sum_{n=0}^{\infty} p_{n,\sigma}(x) \cdot \bar{z}^n e^{\sigma_n \bar{z}}, \qquad p_{n,\sigma}(x) = \int_{\sigma_0}^{x} \int_{\sigma_1}^{t_1} \cdots\cdots \int_{\sigma_{n-1}}^{t_{n-1}} dt_n dt_{n-1} \cdots\cdots dt_1,$$

which converges absolutely and uniformly for $|z| \leqq r_0 < \frac{1}{el}$.

Again let us put

$$\bar{z} = \frac{1}{\zeta} \quad \text{and} \quad \sigma_n = \sigma a_n, \quad (\sigma > 0,\ n = 0, 1, 2, \ldots\ldots)$$

Then (2. 1) becomes as follows:

$$(2.\ 2)\qquad \frac{1}{\zeta} e^{\frac{x}{\zeta}} = \sum_{n=0}^{\infty} p_{n,\sigma}(x) \frac{1}{\zeta^{n+1}} e^{\frac{\sigma a_n}{\zeta}}$$

which converges absolutely and uniformly for $|\zeta| \geqq r' > el$ (when $l = 0$, r' can take any finite value).

If $f(z)$ be an integral transcendental function of type σ and of the first order, the function defined by

$$f^*(z) = f\left(\frac{z}{\sigma}\right)$$

is an integral transcendental function of type 1 and of the first order.

Therefore if we put

$$f^*(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n,$$

and
$$\psi(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad f^*(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \psi(\zeta) \frac{1}{\zeta} e^{\frac{z}{\zeta}} d\zeta, \qquad (r < 1).$$

we can easily show that $\psi(z)$ is regular and analytic for $|z| < 1$.

Since $el < 1$, we can take $r' = 1 - \delta < el$ (δ being a positive constant < 1).

Now, multiplying the both sides of (2. 2) by $\frac{1}{2\pi i}\psi(\zeta)$ and integrating term by term, we get

$$(2.\ 3)\quad f^*(x) = \sum_{n=0}^{\infty} p_{n,\sigma}(x) \cdot \frac{1}{2\pi i} \int_{|\zeta|=1-\delta} \psi(\zeta) \frac{1}{\zeta^{n+1}} e^{\frac{\sigma a_n}{\zeta}} d\zeta = \sum_{n=0}^{\infty} f^{*(n)}(\sigma a_n) p_{n,\sigma}(x)$$

which converges absolutely for any finite value of $|x|$.

On the other hand we have, putting $x=\sigma z$,

$$f^*(\sigma z)=f(z)\,, \quad f^{*(n)}(\sigma\alpha_n)=\frac{1}{\sigma^n}f^{(n)}(\alpha_n)\,, \quad (n=0,1,2,\ldots\ldots). \tag{2. 4}$$

and moreover we can easily show that

$$p_{n,\sigma}(\sigma z)=\int_{\sigma\alpha_0}^{\sigma z}\int_{\sigma\alpha_1}^{t_1}\cdots\cdots\int_{\sigma\alpha_{n-1}}^{t_{n-1}} dt_n dt_{n-1}\cdots\cdots dt_1=\sigma^n p_n(z)\,, \quad (n=0,1,2,\ldots\ldots). \tag{2. 5}$$

From (2. 3), (2. 4) and (2. 5) we can conclude that

THEOREM II. *Let $\{\alpha_n\}$ be a set of points such that*

$$\overline{\lim_{n=\infty}}|\alpha_n|=L<\frac{1}{e\sigma}\,, \qquad (\sigma>0)$$

Then any integral transcendental function of type σ and of the first order can be uniquely expanded into the series of the form:

$$f(z)=\sum_{n=0}^{\infty}f^{(n)}(\alpha_n)\cdot p_n(z)$$

which converges absolutely and uniformly for any finite domain of z.[1]

From this theorem, it follows that

THEOREM III. *Let $f(z)$ be an integral transcendental function of type σ and of the first order, and let α_n be a zero of $f^{(n)}(z)$.*

Then if

$$\overline{\lim_{n=\infty}}|\alpha_n-z_0|=L<\frac{1}{e\sigma}\,,$$

$f(z)$ should vanish identically, where z_0 is a fixed point.

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1) The generalization of this theorem for a regular function in $|z|<R$ and for an integral transcendental function of any type and of any order will be given in my paper which will appear in another place.