62. A Generalization of Ostrowski's Theorem on "Overconvergence" of Power Series.

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In my previous paper,[1] I have proved that a function $f(z)$, regular and analytic for $|z|<r(r>1)$, can be expanded into the series of the form

(1) $$f(z)=\sum_{n=0}^{\infty} c_n z^n e^{\bar{\alpha}_n z}$$

which converges absolutely and uniformly for $|z|\leqq 1$ provided that

$$\overline{\lim_{n=\infty}}|\alpha_n|=L<\frac{1}{e}.$$

Let $\{\pi_n(z)\}$ be a sequence of polynomials defined by

$$p_n(z)=\frac{1}{2\pi}\int_{|\zeta|=1}\pi_n(\zeta)e^{z\bar{\zeta}}|d\zeta|, \qquad (n=0,\,1,\,2,\,\ldots\ldots)$$

where $$p_0(z)=1\,, \quad p_n(z)=\int_{\alpha_0}^{z}\int_{\alpha_1}^{t_1}\cdots\cdots\int_{\alpha_{n-1}}^{t_{n-1}}dt_n dt_{n-1}\cdots\cdots dt_1\,, \qquad (n\geqq 1)\,.$$

Since $\{\pi_n(z)\}$ and $\{z^n e^{\bar{\alpha}_n z}\}$ are each other biorthogonal[2] on $|z|=1$, we have, from (1),

$$\frac{1}{1-\bar{x}z}=\sum_{n=0}^{\infty}\overline{\pi_n(x)}\,z^n e^{\bar{\alpha}_n z}, \qquad (|x|<1,\ |z|<\frac{1}{|x|})$$

or

(2) $$\frac{1}{\zeta-x}=\sum_{n=0}^{\infty}\pi_n(x)\frac{1}{\zeta^{n+1}}e^{\frac{\alpha n}{\zeta}}, \qquad (|x|<1,\ |\zeta|>|x|)$$

the series on the right hand side of (2) being convergent absolutely and uniformly for $|\zeta|\geqq r'>|x|$.

Now let $f(z)$ be a function, regular and analytic for $|z|<1$, with at least one singular point on $|z|=1$. Then the function defined by

$$F(z)=\frac{1}{2\pi i}\int_{|\zeta|<1}f(\zeta)\frac{1}{\zeta}e^{\frac{z}{\zeta}}d\zeta$$

may easily be shown to be an integral transcendental function of type 1 and of the first order, and this can be uniquely determined if

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1) S. Takenaka: On the expansion of an integral transcendental function of the first order in generalized Taylor's series, Proc., 8 (1932), 59.

2) See Takenaka loc. cit.

$$F^{(n)}(a_n)=\frac{1}{2\pi i}\int_{|\zeta|<1} f(\zeta)\frac{1}{\zeta^{n+1}}e^{\frac{a_n}{\zeta}}d\zeta\,, \qquad (n=0,\,1,\,2,\,\ldots\ldots)$$

are given. (See my paper loc. cit. [1]).

Multiplying both sides of (2) by $\frac{1}{2\pi i}f(z)$ and integrating term by term we get (putting z in place of x)

$$(3) \qquad f(z)=\sum_{n=0}^{\infty}c_n\pi_n(z)\,, \qquad c_n=F^{(n)}(a_n)\,, \qquad (n=0,\,1,\,2,\,\ldots\ldots)$$

which converges absolutely and uniformly for $|z|\leqq r<1$.

Particularly if we put

$$a_n=-a\,, \qquad (n=0,\,1,\,2,\,\ldots\ldots)\,,$$

it can easily be shown that (3) holds good for any finite value of $|a|$.

In this special case, we have

$$p_n(z)=\frac{1}{n!}(z+a)^n\,, \qquad (n=0,\,1,\,2,\,\ldots\ldots)$$

and
$$\pi_n(z)=z^n\sum_{\nu=0}^{n}\frac{1}{\nu!}\left(\frac{a}{z}\right)^{\nu}\,, \qquad (n=0,\,1,\,2,\,\ldots\ldots)\,.$$

Since
$$\lim_{n=\infty}\frac{\pi_n(z)}{z^n}=e^{\frac{a}{z}}\,, \qquad (z\neq 0)\,,$$

we have
$$\lim_{n=\infty}\sqrt[n]{|\pi_n(z)|}=|z|\,.$$

Moreover, by the definition of the type and the order of integral transcendental functions, we have

$$\overline{\lim_{n=\infty}}\sqrt[n]{|F^{(n)}(-a)|}=1\,.$$

Therefore the series

$$(4) \quad f(z)=\sum_{n=0}^{\infty}F^{(n)}(-a)\pi_n(z)\,, \quad \pi_n(z)=z^n\sum_{\nu=0}^{n}\frac{1}{\nu!}\left(\frac{a}{z}\right)^{\nu}, \quad (n=0,\,1,\,2,\,\ldots\ldots)$$

must absolutely converge for $|z|<1$ and not converge for $|z|>1$.

By the use of (4) we can prove the following theorem:

Theorem I. *If in* $f(z)=\sum_{\nu=0}^{\infty}a_\nu\pi_{\lambda_\nu}(z)$, $\overline{\lim_{n=\infty}}\sqrt[n]{|a_n|}=1$, *there are infinitely many suffixes such that*

$$\lambda_{\nu_k+1}-\lambda_{\nu_k}>\theta\lambda_{\nu_k}\,, \qquad (\theta>0)\,,$$

then the series $\sum_{k=0}^{\infty}\Delta_k(z)$ *where*

$$\Delta_0(z)=a_0+\cdots\cdots+a_{\nu_1}\pi_{\lambda_{\nu_1}}(z)\,, \quad \ldots\ldots\,,$$

$$\Delta_k(z)=a_{\nu_k+1}\pi_{\lambda_{\nu_k+1}}(z)+\cdots\cdots+a_{\nu_{k+1}}\pi_{\lambda_{\nu_{k+1}}}(z)\,,$$

converges uniformly in the neighbourhood of every regular point on the circle of convergence $|z|=1$.

PROOF. Putting $z=xe^{i\theta}$, $\alpha=\beta e^{i\theta}$ and

$$F(z)=F(xe^{i\theta})=\phi(x)\,,$$

we have $$\sum_{n=0}^{\infty}F^{(n)}(-\alpha)z^n\sum_{\nu=0}^{n}\frac{1}{\nu!}\left(\frac{\alpha}{z}\right)^{\nu}=\sum_{n=0}^{\infty}\phi^{(n)}(-\beta)x^n\sum_{\nu=0}^{n}\frac{1}{\nu!}\left(\frac{\beta}{x}\right)^{\nu}$$

in which $\phi(x)$ is also an integral transcendental function of type 1 and of the first order. Thus without any loss of generality we prove the theorem for $z=1$.

As usual we put

$$R_n(z)=f(z)-\sum_{k=0}^{n-1}\Delta_k(z)$$

and take three circles with centers at $z=\frac{1}{2}$ and radii r_1, r_2, r_3 such

that $$\frac{1}{2}+r_1<\frac{1}{2}+r_2<1<\frac{1}{2}+r_3$$

and $f(z)$ is regular in and on these circles.

Applying Hadamard's theorem of three circles, we have

$$\log\frac{r_3}{r_1}\log M_2^{(n)}\leqq\log\frac{r_3}{r_2}\log M_1^{(n)}+\log\frac{r_2}{r_1}\log M_3^{(n)}$$

where $$M_k^{(n)}=\max_{|\zeta-\frac{1}{2}|=r_k}|R_n(z)|\,,\qquad (k=1,2,3)\,.$$

We introduce another circle with center at $\frac{1}{2}$ and radius r_1' such that

$$\frac{1}{2}+r_1<\frac{1}{2}+r_1'<1\,.$$

Furthermore let us put

$$\tfrac{1}{2}+r_1=1-\delta\rho\,,\quad \tfrac{1}{2}+r_1'=1-\delta^2\,,\quad \tfrac{1}{2}+r_2=1+\delta^2\,,\quad \tfrac{1}{2}+r_3=1+\delta\rho$$

where ρ is a fixed positive number.

Since $$|F^{(n)}(-\alpha)|=\left|\frac{1}{2\pi i}\int_{|\zeta|=1-\delta^2}f(\zeta)\frac{1}{\zeta^{n+1}}e^{-\frac{\alpha}{\zeta}}d\zeta\right|$$

$$\leqq Se^{\frac{|\alpha|}{1-\delta^2}}\frac{1}{(1-\delta^2)^{n+1}}<S'\frac{1}{(1-\delta^2)^n}$$

in which $$S=\max_{|z|=1-\delta^2}|f(z)|$$

and S' is a positive constant depending only on δ, we have

$$M_1^{(n)} \leqq \sum_{\nu=\nu_n+1}^{\infty} |F^{(\lambda_\nu)}(-a)\pi_{\lambda_\nu}(z)| < \sum_{\nu=\nu_n+1}^{\infty} |F^{(\lambda_\nu)}(-a)z^{\lambda_\nu}e^{\left|\frac{a}{z}\right|}|$$

$$< S'e^{\frac{|a|}{1-\delta\rho}} \sum_{\nu=\nu_n+1}^{\infty} \left(\frac{1-\delta\rho}{1-\delta^2}\right)^{\lambda_\nu} < S_1\left(\frac{1-\delta\rho}{1-\delta^2}\right)^{\lambda_{\nu_n+1}}$$

where S_1 is a positive constant depending only on δ.

If we put $M = \max_{|z-\frac{1}{2}|=r_3} |f(z)|$, we have

$$M_3^{(n)} \leqq M + \sum_{\nu=0}^{\nu_n} |F^{(\lambda_\nu)}(-a)\pi_{\lambda_\nu}(z)| < M + S'e^{\frac{|a|}{1+\delta\rho}} \sum_{\nu=0}^{\nu_n} \left(\frac{1+\delta\rho}{1-\delta^2}\right)^{\lambda_\nu}$$

$$< S_2\left(\frac{1+\delta\rho}{1-\delta^2}\right)^{\lambda_{\nu_n}},$$

again S_2 being a positive constant which depends only on δ.

Therefore we have

$$\log\frac{r_3}{r_1}\log M_2^{(n)} < \log\frac{\frac{1}{2}+\delta\rho}{\frac{1}{2}+\delta^2}\log\left(\frac{1-\delta\rho}{1-\delta^2}\right)^{\lambda_{\nu_n+1}}$$

$$+\log\frac{\frac{1}{2}+\delta^2}{\frac{1}{2}-\delta\rho}\log\left(\frac{1+\delta\rho}{1-\delta^2}\right)^{\lambda_{\nu_n}} + S_3$$

from which, by a similar discussion as in Bieberbach's Funktionentheorie Bd. II, 295, we can prove Theorem I.

From Theorem I, we can conclude that

THEOREM II. *Let $f(z)$ be regular and analytic for $|z|<1$ and let*

$$F(z) = \frac{1}{2\pi i}\int_{|\zeta|<1} f(\zeta)\frac{1}{\zeta}e^{\frac{z}{\zeta}}d\zeta$$

be an integral transcendental function of type 1 and of the first order.

If $F^{(n)}(-a)=0$ for $\lambda_{\nu+1}>n>\lambda_\nu$

and $F^{(n)}(-a)\neq 0$ for $n=\lambda_\nu$

where $\lambda_\nu(\nu=0, 1, 2\ldots\ldots)$ are integers such that

$$\lambda_{\nu+1}-\lambda_\nu > \theta\lambda_\nu, \qquad \theta>0$$

Then $|z|=1$ is a natural cut for $f(z)$. Here a is any complex constant.

This is a generalization of a theorem of Hadamard's.