96. Theory of Connections in the Generalized Finsler Manifold. II.

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As a continuation of the previous paper[1] we shall now define a more general connection than the precedings and show that all kinds of connections, which are known up to date and seem independent from each other, are its special cases. The theories of these various connections will be now systematized and unified under this unique theory.

1. At every point of an n-dimensional manifold M_n with a coordinate system x^ν ($\nu=1, 2, \ldots\ldots, n$), we consider its tangent n-dimensional affine flat manifold E_n. In every manifold E_n let us denote with

$$(1.1) \qquad \underset{(a)}{\overset{i}{p}} = \underset{(a)}{\overset{i}{p}}{}^{\nu_1\ldots\nu_{\alpha_a}}_{\nu_1\ldots\nu_{\beta_a}} \qquad \begin{array}{l} i=0, 1, 2, \ldots\ldots, r \\ a=1, 2, \ldots\ldots, a_i \end{array}$$

pseudo affinors of class i and of order $\alpha_a+\beta_a$, which have α_a contravariant and β_a covariant indices, where i, a_i, α_a and β_a are all integers.

We adjoin to every point of the manifold M_n a system of undetermined affinors $\underset{(a)}{\overset{i}{p}}$ belonging to its tangent manifold E_n at that point, and take every system of a point x^ν in M_n and $\sum_i^{0,r} a_i$ number of affinors $\underset{(a)}{\overset{i}{p}}$ adjoined to that point x^ν as a new element (x, p). The *generalized* manifold $\mathfrak{F}_n$ means here the totality of all such elements (x, p).

2. We adjoin next to every element (x, p) of the generalized manifold $\mathfrak{F}_n$ an N-dimensional affine flat manifold E_N, whose coordinate system X^Λ ($\Lambda=1, 2, \ldots\ldots, N$) be transformed linearly by a coordinate transformation of the manifold M_n $x^\nu \to x'^\nu$:

$$(2.1) \qquad X'^\Lambda = \mathfrak{P}^\Lambda_M X^M,$$

where $\mathfrak{P}^\Lambda_M$'s are functions of the element (x, p) of the generalized manifold $\mathfrak{F}_n$. Let

$$(2.2) \qquad \underset{(A)}{\overset{I}{P}} = \underset{(A)}{\overset{I}{P}}{}^{M_1\ldots M_{\alpha_A}}_{\Lambda_1\ldots\Lambda_{\beta_A}} \qquad \begin{array}{l} I=0, 1, 2, \ldots\ldots, R \\ A=1, 2, \ldots\ldots, A_I \end{array}$$

be a system of un-determined pseudo affinors of class I and of order $\alpha_A+\beta_A$, which have α_A contravariant and β_A covariant indices and are belonging to the manifold E_N, then we consider the totality $\mathfrak{M}_n$ of all

1) A. Kawaguchi: Theory of connections in the generalized Finsler manifold, Proc. 7 (1931), 211–214.

systems (x, p, P) of an element (x, p) and $\sum_{I}^{0,R} A_I$ number of affinors ${}_{(A)}\overset{I}{P}$ in the manifold E_N adjoined to that element (x, p). $\mathfrak{M}_n$ will be called the *hyper-generalized manifold.*

3. In order to bring to light the distribution of the affinors ${}_{(a)}\overset{i}{p}$ and ${}_{(A)}\overset{I}{P}$ in all points the manifold M_n, which we can determine arbitrarily, we introduce the following *base connections*, which are not fixed in general, as those in the previous paper,[1)]

$$ {}_{(a)}\overset{i}{\delta}_{(a)}\overset{i}{p}{}^{\nu_1\ldots\nu_{\alpha_a}}_{\lambda_1\ldots\lambda_{\beta_a}} = d_{(a)}\overset{i}{p}{}^{\nu_1\ldots\nu_{\alpha_a}}_{\lambda_1\ldots\lambda_{\beta_a}} + {}_{(a)}\overset{i}{\Gamma}{}^{\nu_1\ldots\nu_{\alpha_a}}_{\lambda_1\ldots\lambda_{\beta_a}\mu}(x, p)dx^\mu , \tag{3.1}$$

$$ {}_{(a)}\overset{i}{\delta}_{(A)}\overset{I}{P}{}^{M_1\ldots M_{\alpha_A}}_{\Lambda_1\ldots\Lambda_{\beta_A}} = d_{(A)}\overset{I}{P}{}^{M_1\ldots M_{\alpha_A}}_{\Lambda_1\ldots\Lambda_{\beta_A}} + {}_{(A)}\overset{I}{\Gamma}{}^{M_1\ldots M_{\alpha_A}}_{\Lambda_1\ldots\Lambda_{\beta_A}\mu}(x, p, P)dx^\mu , \tag{3.2}$$

where ${}_{(a)}\overset{i}{\Gamma}$'s are functions of the element of the generalized manifold $\mathfrak{F}_n$ in general and ${}_{(A)}\overset{I}{\Gamma}$'s functions of the element of the hyper-generalized manifold $\mathfrak{M}_n$. Put

$$ {}_{(a)}\overset{i}{\varphi}{}^{\nu_1\ldots\nu_{\alpha_a}}_{\lambda_1\ldots\lambda_{\beta_a}}(x, p) = {}_{(a)}\overset{i}{\delta}_{(a)}\overset{i}{p}{}^{\nu_1\ldots\nu_{\alpha_a}}_{\lambda_1\ldots\lambda_{\beta_a}} - {}_{(a)}\overset{i}{\Gamma}{}^{\nu_1\ldots\nu_{\alpha_a}}_{\lambda_1\ldots\lambda_{\beta_a}\mu}(x, p)dx^\mu , \tag{3.3}$$

$$ {}_{(A)}\overset{I}{\psi}{}^{M_1\ldots M_{\alpha_A}}_{\Lambda_1\ldots\Lambda_{\beta_A}}(x, p, P) = {}_{A}\overset{I}{\delta}_{(a)}\overset{I}{P}{}^{M_1\ldots M_{\alpha_A}}_{\Lambda_1\ldots\Lambda_{\beta_A}} - {}_{(A)}\overset{I}{\Gamma}{}^{M_1\ldots M_{\alpha_A}}_{\Lambda_1\ldots\Lambda_{\beta_A}\mu}(x, p, P)dx^\mu , \tag{3.4}$$

then (3.1) and (3.2) become

$$ d_{(a)}\overset{i}{p}{}^{\nu_1\ldots\nu_{\alpha_a}}_{\lambda_1\ldots\lambda_{\beta_a}} = {}_{(a)}\overset{i}{\varphi}{}^{\nu_1\ldots\nu_{\alpha_a}}_{\lambda_1\ldots\lambda_{\beta_a}}(x, p) \quad \text{and} $$
$$ d_{(A)}\overset{I}{P}{}^{M_1\ldots M_{\alpha_A}}_{\Lambda_1\ldots\Lambda_{\beta_A}} = {}_{(A)}\overset{I}{\psi}{}^{M_1\ldots M_{\alpha_A}}_{\Lambda_1\ldots\Lambda_{\beta_A}}(x, p, P) , \tag{3.5}$$

but in this case the functions ${}_{(a)}\overset{i}{\varphi}$ and ${}_{(A)}\overset{I}{\Psi}$ are not uniquely determined in general and we can determine these functions arbitrarily in every case.[2)] If we determine once these functions and adjoin one and only one system of affinors ${}_{(a)}\overset{i}{p}$ and ${}_{(A)}\overset{I}{P}$ to an arbitrary point of the manifold M_n, then the distribution of the affinors ${}_{(a)}\overset{i}{p}$ and ${}_{(A)}\overset{I}{P}$ will be determined in all points of the manifold M_n, where the functions ${}_{(a)}\overset{i}{\varphi}$ and ${}_{(A)}\overset{I}{\Psi}$ are regular analytic.

4. Consider a field of contravariant or covariant vector v^M or w_Λ belonging to the manifold E_N, whose components are functions of the

1) Loc. cit.

2) See A. Kawaguchi: Die Differentialgeometrie in der verallgemeinerten Mannigfaltigkeit, which will be shortly published in the Rendiconti del Circolo Matematico di Palermo.

element (x, p, P) of the hypor-generalized manifold $\mathfrak{M}_n$; then we define a new connection for such vector fields:

(4. 1) $$\delta v^M = dx^\mu \nabla_\mu v^M = dv^M + \Gamma^M(v, x, p, P),$$

(4. 1) $$\delta w_\Lambda = dx^\mu \nabla_\mu w_\Lambda = dw_\Lambda - \Gamma'_\Lambda(w, x, p, P),$$

where Γ^M's as well as Γ'_Λ's depend upon the vector v^M or w_Λ and the element (x, p, P). The parameters Γ^M and Γ'_Λ are transformed by (2. 1) as follows:

(4. 3) $$\mathfrak{P}^M_\Lambda \Gamma^\Lambda - (d\mathfrak{P}^M_\Lambda) v^\Lambda = \bar{\Gamma}^M,$$

(4, 4) $$\mathfrak{Q}^M_\Lambda \Gamma'_M - (d\mathfrak{Q}^M_\Lambda) w_M = \bar{\Gamma}'_\Lambda,$$

where $$\mathfrak{P}^N_M \mathfrak{Q}^M_\Lambda = \delta^N_\Lambda .$$

Γ^M and Γ'_Λ must be homogeneous of one dimension with regard to v^M and w_Λ respectively:

(4. 5) $$\Gamma^M = \frac{\partial \Gamma^M}{\partial v^\Lambda} v^\Lambda, \qquad \Gamma'_\Lambda = \frac{\partial \Gamma'_\Lambda}{\partial w_M} w_M$$

and of zero dimension with regard to ${}_{(a)}\overset{0}{p}$'s and ${}_{(A)}\overset{0}{P}$'s. Moreover they are invariant for any variation of the pseudo affinors: ${}_{(a)}\overset{i}{p}{}' = \sigma^i{}_{(a)}\overset{i}{p}$ and ${}_{(A)}\overset{I}{P}{}' = \rho^I{}_{(A)}\overset{I}{P}$. The covariant differential and derivatives of a vector can be written down easily from (4. 1) or (4. 2) by means of (3. 5) and (3. 1) or (3. 2):

(4. 6)
$$\delta v^M = \frac{\partial v^M}{\partial x^\mu} dx^\mu + \Gamma^M + \sum_i \sum_a \frac{\partial v^M}{\partial\, {}_{(a)}\overset{i}{p}{}^{\nu_1 \ldots \nu_{\alpha_a}}_{\lambda_1 \ldots \lambda_{\beta_a}}} {}_{(a)}\overset{i}{\varphi}{}^{\nu_1 \ldots \nu_{\alpha_a}}_{\lambda_1 \ldots \lambda_{\beta_a}} + \sum_I \sum_A \frac{\partial v^M}{\partial\, {}_{(A)}\overset{I}{P}{}^{M_1 \ldots M_{\alpha_A}}_{\Lambda_1 \ldots \Lambda_{\beta_A}}} {}_{(A)}\overset{I}{\psi}{}^{M_1 \ldots M_{\alpha_A}}_{\Lambda_1 \ldots \Lambda_{\beta_A}},$$

(4. 7)
$$\delta w_\Lambda = \frac{\partial w_\Lambda}{\partial x^\mu} dx^\mu - \Gamma'_\Lambda + \sum_i \sum_a \frac{\partial w_\Lambda}{\partial\, {}_{(a)}\overset{i}{p}{}^{\nu_1 \ldots \nu_{\alpha_a}}_{\lambda_1 \ldots \lambda_{\beta_a}}} {}_{(a)}\overset{i}{\varphi}{}^{\nu_1 \ldots \nu_{\alpha_a}}_{\lambda_1 \ldots \lambda_{\beta_a}} + \sum_I \sum_A \frac{\partial w_\Lambda}{\partial\, {}_{(A)}\overset{I}{P}{}^{M_1 \ldots M_{\alpha_A}}_{\Lambda_1 \ldots \Lambda_{\beta_A}}} {}_{(A)}\overset{I}{\psi}{}^{M_1 \ldots M_{\alpha_A}}_{\Lambda_1 \ldots \Lambda_{\beta_A}};$$

(4. 8)
$$\nabla_\mu v^M = \frac{\partial v^M}{\partial x^\mu} + \Gamma^M_\mu + \sum_i \sum_a \frac{\partial v^M}{\partial\, {}_{(a)}\overset{i}{p}{}^{\nu_1 \ldots \nu_{\alpha_a}}_{\lambda_1 \ldots \lambda_{\beta_a}}} {}_{(a)}\overset{i}{\varphi}{}^{\nu_1 \ldots \nu_{\alpha_a}}_{\lambda_1 \ldots \lambda_{\beta_a} \mu} + \sum_I \sum_A \frac{\partial v^M}{\partial\, {}_{(A)}\overset{I}{P}{}^{M_1 \ldots M_{\alpha_A}}_{\Lambda_1 \ldots \Lambda_{\beta_A}}} {}_{(A)}\overset{I}{\psi}{}^{M_1 \ldots M_{\alpha_A}}_{\Lambda_1 \ldots \Lambda_{\beta_A} \mu},$$

(4. 9)
$$\nabla_\mu w_\Lambda = \frac{\partial w_\Lambda}{\partial x^\mu} - \Gamma'_{\Lambda\mu} + \sum_i \sum_a \frac{\partial w_\Lambda}{\partial\, {}_{(a)}\overset{i}{p}{}^{\nu_1 \ldots \nu_{\alpha_a}}_{\lambda_1 \ldots \lambda_{\beta_a}}} {}_{(a)}\overset{i}{\varphi}{}^{\nu_1 \ldots \nu_{\alpha_a}}_{\lambda_1 \ldots \lambda_{\beta_a} \mu} + \sum_I \sum_A \frac{\partial w_\Lambda}{\partial\, {}_{(A)}\overset{I}{P}{}^{M_1 \ldots M_{\alpha_A}}_{\Lambda_1 \ldots \Lambda_{\beta_A}}} {}_{(A)}\overset{I}{\psi}{}^{M_1 \ldots M_{\alpha_A}}_{\Lambda_1 \ldots \Lambda_{\beta_A} \mu},$$

where $$\Gamma^M_\mu dx^\mu = \Gamma^M, \qquad \Gamma'_{\Lambda\mu} dx^\mu = \Gamma'_\Lambda,$$

and $${}_{(a)}\overset{i}{\varphi}{}^{\nu_1 \ldots \nu_{\alpha_a}}_{\lambda_1 \ldots \lambda_{\beta_a} \mu} = {}_{(a)}\nabla_\mu\, {}_{(a)}\overset{i}{p}{}^{\nu_1 \ldots \nu_{\alpha_a}}_{\lambda_1 \ldots \lambda_{\beta_a}} - {}_{(a)}\overset{i}{\Gamma}{}^{\nu_1 \ldots \nu_{\alpha_a}}_{\lambda_1 \ldots \lambda_{\beta_a} \mu},$$

$$ {}_{(A)}\overset{I}{\psi}{}^{M_1\ldots M_{\alpha_A}}_{\Lambda_1\ldots\Lambda_{\beta_A}\mu} = {}_{(A)}\overset{I}{\nabla}_{\mu\,(A)}\overset{I}{P}{}^{M_1\ldots M_{\alpha_A}}_{\Lambda_1\ldots\Lambda_{\beta_A}} - {}_{(A)}\overset{I}{\Gamma}{}^{M_1\ldots M_{\alpha_A}}_{\Lambda_1\ldots\Lambda_{\beta_A}\mu}. $$

5. In order to derive some special connections, we assume in this section that the manifolds E_n and E_N coincide with each other.

(i) It is clear that our general connection contains the so-called non-linear connection[1] as its special case.

(ii) For $a_i=1$, $\alpha_a=1$, $\beta_a=0$ and $\overset{i+1}{p}{}^{\nu}=\delta\overset{i}{p}{}^{\nu}$, we get the connection in the previous paper.[2]

(iii) For $r=0$ and $a_0=2$, we consider a pair of vectors v^{ν} and w_{λ} with the relation $v^{\nu}w_{\lambda}=0$ and put $v^{\nu}={}_{(1)}\overset{0}{p}{}^{\nu}$, $w_{\lambda}={}_{(2)}\overset{0}{p}_{\lambda}$, since we can choose ${}_{(a)}\overset{i}{p}$ arbitrarily in every case. Moreover we take the parameters of the connection

$$ \Gamma^{\nu}(v,w)=\left(\frac{\partial\varphi_{\mu}}{\partial w_{\nu}}-r_{\mu}v^{\nu}\right)dx^{\mu} \quad\text{and}\quad \Gamma'_{\lambda}(v,w)=\left(\frac{\partial\varphi_{\mu}}{\partial v^{\lambda}}-s_{\mu}w_{\lambda}\right)dx^{\mu}, $$

where φ_{μ}'s must be homogeneous of one dimension and r_{μ} and s_{μ} of zero dimension with regard to v^{ν} as well as w_{λ} by our assumption. This is the connection, which has been derived by Wirtinger.[3] From this standpoint we can derive the *generalized Wirtinger connection* too, which is applied to any pair of arbitrary vectors.

(iv) For $i=1$, $a_i=s$, $\alpha_a=0$ and $\beta_a=1$, we have a connection which depends upon s directions ${}_{(1)}\overset{1}{p}_{\nu}, \ldots\ldots, {}_{(s)}\overset{1}{p}_{\nu}$. By putting ${}_{(a)}\overset{1}{p}_{\nu}=\dfrac{\partial_{(a)}\varphi}{\partial x^{\nu}}$, this connection becomes what depends upon a manifold of $n-s$ dimensions ${}_{(a)}\varphi(x)=0$ $(a=1,2,\ldots\ldots,s)$ spreaded in the manifold M_n. This is also an extension of the connection of the Finsler space, which depends upon a curve only.

(v) We can derive many other interesting connections, for example, what depends upon a part of a manifold of any dimensions or upon an algebraic manifold of any order and of any dimensions.

6. The general linear connection of König[4] is clearly a special case of our connection too. Accordingly we can derive various connections from ours, which are extensions of projective and conformal connections.[5]

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1) H. Friesecke: Math. Annalen, **93** (1925), 101–118; E. Bortolotti: Annals of Math., 2nd series, **32** (1931), 361–377.

2) Loc. cit.

3) W. Wirtinger: Transactions Philos. Soc. Cambridge, **22** (1922), 439–448 and Abhandlungen Hamburg, **4** (1925), 178–200.

4) R. König: Jahresberichte d. Deut. Math. Vereinig., **28** (1920), 213–228.

5) See J. A. Schouten: Rendiconti del Circolo Mat. di Palermo, **50** (1926), 142–169.